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Developing Facility in Solving Verbal Problems

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THE IMPROVEMENT OF pupil achievement in verbal problem solving is an important objective of most upper grade arithmetic teachers. That this objective is not often reached with any degree of satisfaction is evident to all students of arithmetic teaching. It is also quite evident to students of arithmetic teaching that, although there are many problem-solving improvement procedures in use, the most widely used procedure is that of just having pupils work problems without specific directions or suggestions. The other problem-solving improvement procedures practically all suggest specific steps for pupils to engage in. There are, then, two broad classes of procedures, the one general (that of just working problems) and the other characterized by use of specific exercises.

In this latter class there are many and widely differing practices. The divergence is probably not due to faith in the value of divergence but to the search for better procedures. The fact is, as stated earlier, that neither the one broad procedure of "just solve problems" nor the many specific problem-solving improvement procedures have produced the results teachers desire, and therefore the search for new (and that means different) procedures continues. In view of the rather long time that instructors have been concerned with problem solving, it is very doubtful whether any one entirely new procedure of merit will turn up. Improvement will, then, most likely be the result of modification and refinement of plans now in use. This paper is concerned with steps which may lead to such refinements and modifications. Specifically, the paper will present a picture of some classroom and textbook practices illustrating the two broad types of problem-solving improvement procedures, followed by an analysis of the situations.

Examples of Classroom Procedures for Improving Problem Solving

Classroom 1

Miss R, the fifth grade teacher in this classroom, is putting into operation the "just solve problems" procedure for improving problem solving. She began by saying, "Open your books to page 124. Try to work the ten problems on that page. If you cannot work a problem or are not very sure of a solution, raise your hand and I'll try to help you." During the ensuing work period Miss R gave assistance to various children. Her assistance consisted primarily of making suggestions,

asking questions, and offering encouragement. She did not solve problems for the pupils.

When several pupils had finished the ten problems, Miss R said, "Let's stop our work for a few moments. All of you now have the answers for some problems and some have all. I'll read the answers. You check. When I get to the ones you haven't done, you may go back to work. For each incorrect answer try another solution for the problem."

After giving the answers, Miss R said, "If at any time you want to know an answer, you may look at the answer list. I'm leaving it here on my desk." She then again went about the room giving assistance to those who needed it.

Classroom 2

Miss D, the fifth grade teacher in this room, is also trying to improve pupil ability to solve verbal arithmetic problems. She began by saying, "Number downward on your paper from 1 through 6. As I read an arithmetic word problem aloud, try to solve it without using paper and pencil. Write only your answer by the number on your paper." She then read the following:

1. For her birthday party, Jane bought 2 bags of candy with 20 pieces in each bag. She plans to serve each person 5 pieces of candy. How many persons can she serve?

After pausing for about 30 seconds to allow the children to think and record an answer, Miss D asked, "Who would like to tell how he thought in solving the problem and then give his answer to see if others agree with him? I will read the problem again."

Johnny explained, "I know that if there are two bags with 20 pieces in each bag, there would be 40 pieces altogether. A group of 5 pieces is to be served to each person, so I divided to find out how many groups of 5 could be made from 40 pieces. My answer is 8."

Sue explained, "I agree with the answer of 8, but I first divided to find out how many groups of 5 can be made from 20 pieces of candy. This is 4. I then multiplied 4 by 2 because there were 2 bags with 4 groups of 5 in each bag."

Miss D then said, "You worked in a different way but arrived at the same answer. I wonder if the other members of the class noted the reasons why you each began in a different way. What did Johnny want to know first that caused him to begin by multiplying 20 pieces by 2?—What did Sue first want to know that caused her to begin by dividing 20 pieces by 5 pieces?" Miss D continued in a like manner with five other word problems to be solved without paper and pencil, each of which was followed by oral pupil presentation of various ways of thinking.

The use of oral or non-pencil-and-paper exercises is the problem-solving improvement suggestion illustrated in Miss D's room.

Classroom 3

The problem-solving improvement procedure which Miss L is using in this classroom emphasizes the use of drawings. She gave the following directions at the beginning of the class period.

"Use a simple drawing or picture to show how you are thinking as you solve each of the two problems I have written on the board." The problems were:

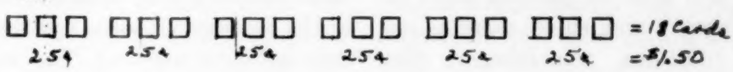
(1) How much will Jane have to pay for 18 Christmas cards if they are being sold at 3 for 25¢?

(2) Mrs. Green bought 4 yards of silk material. She plans to use $1\frac{1}{4}$ yards to make Ann a blouse and $2\frac{3}{8}$ yards to make a blouse for herself. How much material will she have left from which to make a silk head scarf for Ann?

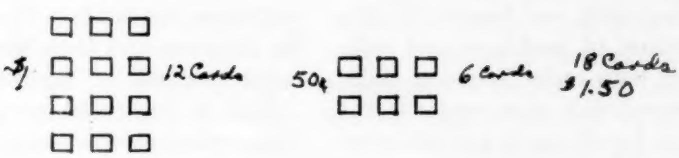
As the children worked on the assignment, Miss L went about the room giving assistance to those who needed it. As the work period progressed, she asked some

pupils to place their drawing solutions on the chalkboard. The drawing solutions below are typical:

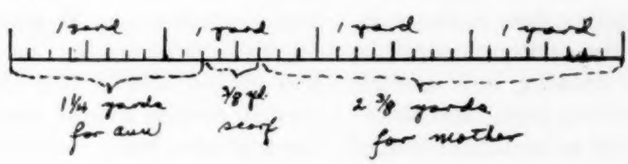
(1a)



(1b)



(2)



In the discussion period that followed, Miss D directed attention to these drawing solutions by such questions as "Can you tell how the person thought who put this solution on the board?"

Classroom 4

In School IV, Miss N and her fifth grade pupils used another procedure as they worked to improve in ability to solve word problems. Four word problems had been written on the chalkboard. Miss N began the class period by saying, "In problem number 1, there are really two questions to be answered, but only one is in written form. Can you find the unwritten question which is hidden from you? Let's do the first problem together. Read it silently."

1. Ann divided 48 roses picked from her mother's rose garden into bouquets of 6 roses each. She sold each bouquet for 50¢. How much money did she receive altogether?

After a pause for reading and consideration, several suggestions were made. The class decided that the hidden question is, "How many bouquets did Ann make?" Miss N then wrote that question on the board and drew an arrow to the place in the original problem where the question might be inserted. She then said, "Try to state the unwritten or hidden number question in each of these other problems on the board and for the ones on page 116 of your book. Write your statement of the question." As the pupils worked, Miss N went about the room giving assistance where it was needed. At the close of the work period various suggested questions were compared.

Analysis of the Procedures

The outstanding features of the instructional practices used in each of these four classrooms follow. In the first classroom, with the emphasis on getting the correct answer, the pupil works primarily alone. All the pupil's time is given to work on his own solution with none or very little time spent in listening to questions, suggestions, and explanations of other pupils. Nor is any time spent in studying (ob-

serving) the solutions of others. Group spirit or group motivation, then, plays a minor role in this classroom.

In the second classroom emphasis is on the answer and also on various ways of solving problems, and since there is no writing of solutions, remembering the thinking done is at a premium. The consideration of the ways of thinking solutions, suggestions, and questions of others makes for class spirit and teamwork. The oral presentation of problems and solutions requires high order listening skills. This oral presentation also results in the pacing of work for all pupils and therefore does not permit fast workers to proceed at a rate that is up to their capacity.

In the third classroom the emphasis is on showing the thinking done in the problem solution, and the asking of pupils to figure out from a drawing how another person thought requires both clear drawing presentations and an understanding of the problem setting as well as its solution. While the assignment made required all pupils to use one type of solution (use of drawings), this type of non-standardized solution almost guarantees a variety of solutions and leaves pupils free to put into practice ingenuity and initiative. The consideration of various drawing solutions makes possible development of class spirit.

In the fourth classroom emphasis is on one specific aspect of problem solving, the formulation of the unstated question in two-step problems. To state this unwritten question calls for an understanding of the problem, and therefore the procedure actually gives more than experience with just a part of the problem-solving process. The specific nature of the assignment makes it easy for pupil and teacher to identify the learning task. The comparison of suggested questions in this procedure results in little disagreement except where errors are made. Therefore, the discussion period is primarily a checking period and does little for the development of class spirit.

In the opinion of the writers, the procedures used in Classroom 1 are definitely inferior to those used in the other classrooms. This conclusion is based on the belief that meeting with success or failure in the attempt to solve a problem contributes little to learning. The student who fails just tries again without benefit of suggestion, analysis from others, or knowledge of errors made. The pupil who succeeds goes no farther. He doesn't explain his thinking and therefore his method is never analyzed or challenged.

Each of the procedures used in the other three classrooms calls for some analysis, and in Classrooms 2 and 3, for presentation of ways of solving, followed by an evaluation. Such analysis, presentation, and evaluation should make for learning for those who fail in the first attempts and should also enable those who succeed to acquire a more thorough grasp of the problems being considered.

Space does not permit further analysis and evaluation of the specific classroom procedures. The presentation, it is believed, has shown the value of looking critically at some specific problem-solving procedures.

To the writers any one of the specific procedures described, but especially the second and the third, seems to offer much better opportunities for developing problem-solving ability than does the general procedure of "just solve problems." It is recommended that most of the instructional time assigned to the improvement of problem solving, then, be devoted to use of specific techniques. It is also recommended that further analysis of actual use of these and other procedures be pursued. Such analysis should then be followed by trial in situations where competent evaluation can be carried out. A study program of this type should result in the identification of the procedures for improving problem solving that should receive the most attention and might even eliminate some of the procedures now recommended.

The "just solve problems" procedure and the "special techniques" for improving problem-solving ability are both present in problem-solving programs found in current arithmetic books. Since the books do not label sets of problems to be used for the "just solve problems" procedure, it is difficult to determine the actual amount of space devoted to this procedure. On the other hand, the various special techniques for improving problem solving and the amount of space devoted to each in textbooks are easy to identify. The types of special problem-solving techniques and the amount of space devoted to each technique in five current fifth grade arithmetics are shown in Table 1.

The facts presented in Table 1 indicate that one may expect to find about five special techniques for improving problem-solving ability used within a single textbook. One may also expect to find an average of about three repetitions or lessons for each type of technique, but a majority of the lesson types are repeated only once or twice in a single book. Problem analysis, designating the process to be used in a solution, and stating the hidden question are the most often used techniques.

The "without pencil and paper" tech-

TABLE 1

TYPES OF TECHNIQUES USED IN FIVE RECENT FIFTH GRADE TEXTBOOKS TO IMPROVE PUPIL ABILITY TO SOLVE VERBAL ARITHMETIC PROBLEMS

	Number of Pages					Total Pages Devoted to Each Technique
	Book A	Book B	Book C	Book D	Book E	
1. Problem Analysis	4	3		2	1	10
2. Writing Original Problems	4					4
3. Designating the Process or Processes for Solving	1			10	1	12
4. Stating the Hidden Question	2	1	2			5
5. Studying Problems without Numbers		1		1		2
6. Two-step Problems with the Two Questions Written		1				1
7. Rewriting a Two-step Problem with Two Questions Written as a Problem with One Written Question		1				1
8. A Written General Reminder that the Problems on the Page Have Two or More Steps				5		5
9. Supplying the Missing Question			1	1		2
10. Supplying the Missing Facts			1			1
11. Without Paper and Pencil			4			4
12. Estimating Answers				4		4
13. Diagrams Drawn for the Pupil to Use in Solving					1	1
14. Directions to Draw a Picture if Needed					1	1
15. Telling Aloud How You Thought in Solving					1	1
16. Solving by More Than One Written Method					2	2
17. Completing a Statement of Rule and Making Up a Simple Problem to Illustrate It					1	1
Total Pages in Each Textbook Using Some Specific Technique for Improving Pupil Ability to Solve Word Problems	11	7	8	23	8	57

nique which was recommended as a worthwhile procedure in the first part of this paper was not even included in four of the five books. Use of drawings in solutions, pupil formulation of problems, and oral explanation of solutions by pupils were not popular in these five textbooks. Still more surprising is the fact that not a single specific procedure for improving problem solving is recommended by all five books.

The sketchy problem-solving improvement program offered by any one textbook, the lack of agreement on procedures among textbooks, and the omission from textbooks of what appear to be promising procedures should present a real challenge to students of arithmetic teaching.

Two Recommendations

In the opinion of the writers, the most significant of these challenges are stated in the two following recommendations:

1. The typical textbook program for improving or developing problem-solving ability has to be supplemented by providing more experience with the techniques recommended and by using promising techniques not included in the textbook.

2. Students of arithmetic teaching need to make studies to determine whether or not proposed problem-solving improvement procedures actually contribute to this ability. The importance of this recommendation is shown by the fact that only one recent research study can be cited in support of any of the seventeen specific techniques listed in Table 1.

EDITOR'S NOTE. All of us would probably agree that each pupil, while he is in school, has a right to the best guidance and instruction that a teacher can give him. The problem lies in finding the *best* for each as an individual. Professor Spitzer and Miss Flournoy have pointed out that we have comparatively little agreement in practice and in textbooks on methods for teaching problem solving. The "just work problems" method may have little value if it stops with "just work problems" but it may be a most fruitful situation for individual help based upon fitting all of the other problem solving devices or methods to each child or sub-group of children at the most opportune time and in the best method. Therein lies the artistry in teaching. Occasionally the best help is the least help that will serve to redirect a pupil's thinking. Too frequently, beginning teachers try to do all of the thinking for their pupils and that defeats the very aim of education. Teachers should study Table 1 carefully and consider when they should use one and when another of the seventeen different procedures. The editor was surprised that so few books suggested the writing of original problems because he has employed that procedure for many years and found it very helpful in having pupils become familiar with different problem-situations and with the use of arithmetic processes in different types of situations. Let us remember that we learn to solve written and oral problems in school so that we may solve them outside of school as well. Spitzer and Flournoy have done a good job in opening this important topic, what shall we do next?

Watching the License Numbers

JESSE OSBORN
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This idea comes from automobile license numbers. It can be carried from the schoolroom to the road.

Write one or more six (or seven) digit numbers on the chalkboard for "getting-settled time" as the pupils enter the room, or for the few minutes before class is dismissed.

One plan is to find how many 7's or multiples of 7 can be found by adding and subtracting digits of the license number. As an example, find the 7's and multiples of 7 for: 836-419.

$8+6$; $3+4$; $6+1$; $6+4-3$; $1+9-3$; $4+1+9$; $8+3+4-1$; $3+6+4+1$; etc.

Any digit, as 9, 8, 6, etc., may be used instead of 7. Another plan is to use two digits at a time to form numbers that are multiples of 7 (or 9, 8, 6, etc.) Use digits of this license number: 6439-152, to form: 14; 21; 35; 42; 49; 56; 63; 91.

Meaning Is the Key

GLADYS RISDEN
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SIX MONTHS AGO these children began to fail in arithmetic:

JUDY, I.Q. 110, got A's in first grade arithemetic but in mid-second she began making "careless" mistakes:

8	8	7	7
5	5	6	6
14	14	12	12

mistakes of 1.

JOE, I.Q. 112, couldn't remember his number facts.

PETE, I.Q. 100, had done well in addition combinations—he did well in subtraction combinations after someone told him to count backward; but how could he remember whether they were to be counted frontward or backward when Teacher mixed them up?

PENNY, I.Q. 115, never finished her work on time. She "diddled" and "day-dreamed," her teacher said.

TED, I.Q. 123, never made a mistake in the number facts but he kept getting "mixed up" in "carrying" and "borrowing" examples. He couldn't get story-problems, either, although he was in the "best" third grade reading group.

Meaning Pays Dividends

Today, six months and some 100 tutoring hours later these five children of high average intelligence are getting A's and B's in arithmetic.

Judy isn't "remembering" combinations. She has a better way—7 and 8: 5 and 10, fifteen. Four threes: two threes and two threes, twelve. 15–6: six out of the ten leaves four, then five more, nine; or 6 and *four* more are ten and *five* more are nine; six and nine are fifteen.

Three months ago Judy was rethinking: three and two, two and two and one more. Seven and three, put two from seven with five and have two fives, ten. She doesn't have to rethink those easy ones now. They just "pop up when she wants them." Seven and eight and four threes will begin "popping up" almost any day now. She isn't using objects to "help me think them" any longer. She is thinking 26 and 8 with objects: 2 dimes, six pennies, 8 more pennies—put 4 pennies with the 6 to make another ten, thirty, four left from the eight, thirty-four. She is almost ready to "think" 26 and 8 without pennies and dimes.

Joe isn't making careless mistakes anymore. When Joe came for tutoring he was counting by quickly tapping his tongue against his teeth. He could tap it seven times without a mistake but when he had to tap it twelve times he slowed down enough to break the co-ordination between saying names of numbers and tapping. And so he said "12" or "10" for eleven taps.

Today Joe doesn't have to "count." He *just knows* all the addition and subtraction combinations and easy multiplication and division facts. He learned them as Judy learned them—regrouping combinations he didn't know into combinations he did know, first with objects and then as seeing helped him develop grouping-sense Joe could think groups without objects.

Joe is learning the hard multiplication and division facts now. He doesn't have to use objects but he does have to write down figures and group them sometimes:

8	8		8
	8		8
8	8		8

Five eights and three eights, forty and twenty-four, sixty-four. He doesn't have to write

$$\begin{array}{r} 40 \\ \underline{24} \end{array}$$

and add with a pencil. Joe has learned to think four tens and two tens as easily as he thinks four ones and two ones. He learned by "seeing" groups of dimes and packages of ten blocks, etc.

Today Pete isn't confusing "addition" with "subtraction." His tutor told him not to use those two words. Instead, she said: "We put groups together or we take them apart. Let's learn to say what we see." So the tutor showed him a group of six blocks then covered three. "I had six. I took three away. There are three left." She showed him five blocks and quickly covered two. "How many did you see first? How many do you see now? Then how many must be under my hand? Now we'll see what I did with figures."

Pete looked to see what his tutor did with blocks and said it with figures. He did all the things possible with six blocks and wrote the facts (saying with figures) as he did them. He "said with figures" all the things he could think of that could be done with six blocks, then he pulled six blocks from the stockpile and checked to see whether there were any he hadn't written.

"Useterbe," says Pete, "I couldn't remember answers. I had to count to find answers, 'nd then I got a different kind an' I couldn't get them 'til I counted backward, 'n then the teacher gave us both kinds all mixed up 'nd I couldn't tell when to count frontward and when to count backward. Now I don't have to count. I just *know*. 's easy now."

Penny never finished her work on time because she, too, was counting to find answers. She didn't make mistakes in counting as Joe did. Her motor co-ordination was better than Joe's and her threshold of fatigue higher but Penny was geared to slower motion. She didn't make fast moves of any kind. Her big toe, which she used to count with, didn't move fast enough to get answers quickly when quantities were big and it was pretty monotonous, sitting there wriggling that big toe all the time and her thoughts did wander.

Penny finishes her work today. She doesn't have to count any more. She too worked at putting groups together, taking them apart, then just thinking them together or thinking them apart, changing groups she didn't know into combinations of groups she did know until she didn't have to think them anymore. They were just there. They just popped out "quick."

Ted isn't making mistakes in carrying and borrowing any more. "Now I have to borrow . . . ", Ted's voice trailed off as he looked at his tutor doubtfully. His tutor said: "Let's find out what we are talking about, then use words to say it that tell what we are doing. I don't like sixty-four dollar words like "borrowing" —not until I really know what they mean—I like common sense words first." So Ted's tutor placed three dimes before him and said, "How much money do you have?"

Ted: "thirdy cents."

Tutor: "Write down what you have."

Ted did so.

"Now give me five ones, five pennies." She was using a word he didn't understand with a word he did understand. Ted thought: "Ones and pennies are the same."

Ted stared at the three dimes and looked up in despair. "I can't."

"I can," said the tutor casually. "I can take a ten (taking a dime) to the bank (the pile of pennies at one side) and trade it for ten ones, ten pennies. Now I have 2 tens and ten ones. Now I can give you five ones." Ted and his tutor did many more "problems" with pennies and dimes, packages of tens and ones (cards and

blocks) and with ten and one-pound sacks of sugar and sand. Then they began "saying with figures." Thereafter for several weeks they did not use terms such as "borrowing" or "carrying"; they talked about trading tens for ones. (borrowing) or having new tens to put with the other tens (carrying).

The day came when Pete exclaimed, "Why I'm borrowing."

"Good," exclaimed his teacher, "You know what that 'sixty-four dollar word' means now. It's all right to use it *after you know what it means.*"

Usefulness Requires Understanding

Thousands of children in our schools today are failing in arithmetic because they can't remember answers or are "careless" or "don't finish the assignments." Thousands are "passing" school arithmetic but are on the road to failure in life-arithmetic because they are "getting answers" by a quick-counting method or because they have good rote memories.

Children who can't remember answers may have impaired apparatus for rote memorizing but they often have superior apparatus for the higher mental processes. Those who are counting have good motor co-ordination and high threshholds of fatigue so they can count even large numbers with co-ordination but they are not seeing groups. Groups are not really groups, to these children; groups are aggregations of ones. Perhaps such children see ones merely because of lack of opportunity and guidance in seeing groups instead of ones, or perhaps there is impairment of their apparatus for seeing groups as groups—for focussing on the group as a whole instead of on one, another one, another one, etc. Guidance and practice will help many children. New brain paths can be developed to bypass the impaired brain tissue that distorts and limits their group-percepts. They can learn to *see* groups, to *think* groups. Soon they say: "Why the answer is right there in my brain. It just pops out." These children can then take their learning upstairs to the higher centers of comparing, abstracting and generalizing, and learn arithmetic for *mastery*, not merely for saying-to-pass-tests.

Children who get processes confused no longer confuse when they have the right guidance and opportunity in putting groups together, in taking apart, and saying what they do in common sense language. Generalizations are the last step in learning, normally. No one can use another man's generalizations effectively and efficiently. Each must have his own. This is a fact of life we often violate from the first day of school arithmetic in all but a very few schools today.

Common Sense Needed

There was once an arithmetic of the counting-house and an arithmetic of the University. The former was "common sense" arithmetic. Here men took groups apart and put them together. They extended their field of thinking by using the abacus to put tens, and tens of tens, together and take them apart.

The University professors, men who found abstract thinking easy, used short-cut language. They didn't say: "Put this ten with these tens." They "carried." They didn't say: "Go over and get a ten and then get change for it," or "trade a ten for ones," "a hundred for tens." They "borrowed" from the next column. They didn't say: "break a quantity into a given number of equal parts, or into equal groups"; they "divided." They didn't change fractions to the same size so they could put them together. They changed fractions to "common denominators," to "higher" and "lower" terms.

They didn't find how many pieces "this size" there are in a piece "that size," they "divided fractions by inverting the divisor and multiplying." They didn't change pieces this size to pieces that size—fourths to eighths, by thinking what a fourth of eight pieces would be and what three-fourths of eight pieces would be. They

divided this denominator by that denominator and multiplied by the numerator. They didn't change hundredths to tenths by breaking this group of pieces into tens since ten hundredths make a tenth. They moved the decimal point over one place to the left.

They, with their love for pigeon-holing, the professors created a filing system of rules—words, but only words to a child. Is it not too bad that our arithmetic books didn't come down from the common sense arithmetic of the counting-house?

Had our arithmetic come from the arithmetic of the counting-house we might be graduating fewer arithmetic-illiterates from our schools today. And there are plenty of A-students among these arithmetic-illiterates. They fooled their teachers into thinking they knew arithmetic because they had good rote memories or because they could count ones accurately and speedily, but they never learned to think groups together and apart. Out of school where problems don't come ticketed and bracketed under rules these arithmetic-illiterates grab for pencil and paper to make every simple computation.

Meaning is the key. The best preparation for the teacher of arithmetic in regular or remedial classes is understanding of the wonderful consistencies and relationships which make mathematics a discipline for handling quantities in life and in the classroom, effectively and economically. This teacher of arithmetic teaches her pupils for *knowing*, not merely for *saying*, if and when she grasps the basic ideas. The only thing we can do with quantities are take them apart and put them together again. We can put them together in unequal groups (addition) and equal groups (multiplication). We take them apart in unequal groups (subtraction) and equal groups (division). We must group—see groups as wholes, not aggregations of ones, if we are to learn for *knowing*, not merely for *saying*.

The well-prepared teacher will find experimenting with thinking groups together and apart, fun. She will say what she is thinking in common sense language, not technical words, until she has made the generalizations which bring the technical words and the ways of stating rules, popping into her head.

A teacher who cannot "say with pieces" $6\frac{1}{2} \div \frac{1}{4}$, who cannot sit down with $6\frac{1}{2}$ "pies" and break them up into "fourths of pies," or take a fourth-of-a-pie and think how much of a third-of-a-pie she is holding—do this until she can think breaking up without seeing the objects, then finally "seeing" her own generalization, seeing why the rule says, "Invert the divisor and multiply"—a teacher who cannot do this and also "do" $\frac{1}{2}$ of $\frac{1}{2}$ and $6 \times \frac{1}{2}$ with pieces, seeing the difference between multiplication and division of fractions, is not ready to teach multiplication of fractions to any pupils, remedial or regular, so that they will learn for *knowing*, not merely for *saying*.

The test of *knowing* is using; *knowing* is more than *saying*.

EDITOR'S NOTE. Gladys Risden is a trained psychologist who has had a great deal of experience in helping pupils who have not been successful with arithmetic in their regular school. She points out that it is not necessarily the lower mental levels that do not learn with meaning and understanding. In fact, the brighter child is often the one who can memorize quickly and "give back" answers to a teacher and who may actually understand very little of what he is saying. She clearly draws the distinction between *knowing* and *saying*. Many students of the teaching of arithmetic are convinced that *knowing* has more lasting qualities and will be the persistent factor for success in a child's later work in school and in life. Memory without understanding is a fleeting thing for most of us. Let us help out pupils to understand, to see, and to know.

Two-Digit Divisors Ending in 4, 5, or 6

Harry E. Benz
Ohio University, Athens

Most teachers find the teaching of division with two-digit divisors a difficult task. The subject has been the occasion for a considerable amount of writing and discussion, and no little research has been carried out in efforts to find ways of organizing the topic for more effective instruction.

Much of the difficulty seems to occur when we are trying to teach children to estimate quotient figures. It is reasonable to assume that readers of this *Journal* are familiar with the issues involved, hence only brief reference will be made to those aspects of the problem which are not germane to the research here reported.

There are several schools of thought on the subject of how children should think when they estimate quotient figures. Most progressive teachers believe that the pupil should understand something about the structure of the number system, especially the significance of the decimal notation and the significance of place value, and they govern their practice accordingly. In the example at the right, for instance, although the pupil may obtain the quotient figure 2 by thinking "How many 4's are there in 9?" or some variant of that phraseology, nobody now believes that he should be permitted to think that he is really simply dividing 9 by 4. Neither should he be permitted to think that the question as stated above just happens to yield a quotient figure which is useful to him in his effort to work the whole example. Some authorities prefer to have the pupil continue to think, "How many 40's are there in 90 (or 93)?" No effort will be made here to evaluate the merits of various ways of thinking, but for present purposes we may assume that the pupil's thinking is: "How many 4's are there in 9?"

$$42\overline{)369}$$

As every teacher knows, using the first digit of the divisor as a "trial divisor" will not always yield the correct quotient figure. Often the "true quotient" will be a smaller number than the one obtained by the process indicated above. When this is so, it is usually one less than the "trial quotient," but not always. Sometimes it differs from the trial quotient by more than one. The exact number of instances in which the true quotient obtained by the process indicated above, which we shall call *Rule A*, is exactly equal to the trial quotient, and the number of instances in which it is one, two, or three less, has been determined and the results are readily available.*

In the example at the right, it will be noted that *Rule A* yields a quotient figure which is not correct, that is, the trial quotient thus obtained is not the true quotient. However, if a number one more than the first digit of the divisor, in this instance 7, is used as the trial divisor, the trial quotient *thus* obtained is the true quotient. This procedure is sometimes referred to as the "in-

$$67\overline{)421}$$

* Jeep, H. A., *Long Division.* A Discussion of Long Division. *Second Yearbook, National Council of Teachers of Mathematics*, p. 41. New York, 1927.

Upton, C. B., "Making Long Division Automatic." *The Teaching of Arithmetic*, p. 251. Tenth Yearbook of the National Council of Teachers of Mathematics, 1935.

Knight, F. B., *Comments on Long Division.* Fourth Yearbook, Department of Superintendence of the National Education Association, p. 208. Washington, 1926.

Grossnickle, F. E., Three articles on long division. *Elementary School Journal*, 32: 299, 442, and 595. Dec. 1931, Feb. 1932, and Apr. 1932.

crease-by-one rule." For purposes of this report it will be referred to as *Rule B*. The exact number of times that *Rule B* will yield the true quotient and the number of times the trial quotient must be adjusted (upward, this time) have also been determined.

It has been noted that *Rule A* works more often when the second digit of the divisor is a small number, and that *Rule B* works more often when the second digit is a large number, although sometimes *Rule A* will work when the divisor ends in 9, and in other instances it will *not* work when the second digit is 1. Similarly, occasionally *Rule B* will work when the divisor ends in 1, and there are instances of its failure to work when the second digit of the divisor is 9. Examples exist in which either rule will work and some in which neither will work.

All this has led to considerable difference of opinion among teachers of arithmetic, writers of textbooks, and other experts in this area, concerning which of the rules (procedures) referred to above should be taught to children. Some advocate teaching children to work all division examples by the use of *Rule A*. Others propose that both rules be used, *Rule A* when the second digit of the divisor is small, and *Rule B* when the units' figure is large. Often these latter persons favor the use of *Rule A* when the divisor ends in 0, 1, 2, 3, or 4, and the use of *Rule B* when the second digit is 5, 6, 7, 8, or 9, although some prefer to treat divisors ending in 0 separately. On the other hand, it would be possible to teach pupils to work all division examples with the use of *Rule B*.

The aim of course is to teach children a procedure or a combination of procedures that will result in obtaining the correct quotient digit as frequently as possible. In the early stages of learning the child should probably write the trial quotient, and then erase and change it when his subsequent steps indicate that it is not correct, although some teachers try to teach children from the beginning to hold the trial quotient in mind without writing it until the necessary subsequent steps have provided information about its correctness. In either case some pedagogical gain would result from the use of procedures which yield the true quotient on the first try as often as possible.

Many authorities think that as children acquire skill in the process of division, and as they acquire additional mental maturity, they should be encouraged to try other procedures which make their computational practices more efficient and more straightforward. It may be hoped that very few readers of this journal have work habits in division which justify the idea that for addition, subtraction, and multiplication one needs only a pencil, but for division one needs both pencil and eraser. Many intelligent adults have, accidentally or otherwise, developed procedures which enable them to write the correct quotient figure at once. Some teachers try to teach pupils such procedures, but not much specific help is available which tells the pupil just exactly how to think the process. In some instances the teacher's instructions do not rise in helpfulness above the level of "Try to get the correct quotient figure the first time."

Special Problems with 4, 5, or 6

Much of the trouble arises in examples in which the second digit in the divisor is in the middle of the decade, that is, when it is 4, 5, or 6. One writer notes that some children in the upper grades use a combination of the two rules, that is they use both *Rule A* and *Rule B*.[1] In some instances both rules will yield the same trial quotient. This then is the true quotient. When the two rules yield different numbers, the true quotient is either one of them, or some number which lies

[1] Morton, R. L., *Teaching Children Arithmetic*. p. 250. Silver, Burdette Company, New York. 1954.

between them. Upton noted[2] that Warren Colburn, in a textbook published in 1822, advocated the use of a procedure such as the one just suggested.

No opinion will be expressed here as to the merits of the proposed procedure. However, it would seem that any such judgment should be based in part on a careful analysis of the numerical situations which are encountered when dividing by two-digit numbers ending in 4, 5, or 6. Such an analysis of a small sample of divisors is presented herewith.

There are 90 two-digit numbers which can be used as divisors, ranging from 10 to 99 inclusive. Some teachers prefer to treat those ending in 0 as a special case, usually simply as an extension of "short" division. Each divisor can be combined with an ascertainable number of dividends, and these dividends, if we let x represent the divisor, range from x to $(10x-1)$ inclusive. The number of possible examples in which each divisor can be involved then is 9 times the value of the divisor. This gives a total of 44,145 examples, if we include as divisors numbers from 10 to 99. In 4050 of these the divisor is a multiple of 10.

There are 27 two-digit divisors ending in 4, 5, or 6. The number of possible examples which can be made with two-digit divisors ending in 4, 5, or 6, is 13,365. This is of course exactly one-third of the total number of possible examples if we omit those in which the divisor ends in 0. Thus it can be seen that the present discussion has possible relevance in one-third of the instances of division where estimation of quotient figures requires some specific procedure beyond that which applies to "short" division.

Teachers who wish to help pupils in the upper grades develop more mature procedures for estimating quotient figures, may well consider the advisability of helping them learn to use a combination of *Rule A* and *Rule B*. One suggestion would be to determine trial quotients by both rules, then put down a figure which seems likely to be the true quotient. Consideration of the value of such a procedure at once raises the following questions:

1. How often do the two "rules" yield the same trial quotient? (When they do, it is the true quotient.)
2. How often do the two rules yield trial quotients which differ by 1?
3. How often do the two rules yield trial quotients which differ by 2? When this occurs, how frequently is the true quotient the number between?

Other questions of a similar nature will occur to the reader. In an effort to provide data for a more intelligent evaluation of the procedure suggested, all the possible examples were analyzed which involve the use of 25, 34, 35, 36, 45, 55, 65, 75, 84, 85, and 86 as divisors. The pertinent data are presented in the accompanying tables. A similar analysis could be made for each of the remaining divisors which end in 4, 5, or 6.

Table I presents information for those two-digit divisors whose second digit is 5. Two divisors, 15 and 95, are included in the table with the knowledge that some teachers prefer not to handle them in the same way they handle the others. In Table I the numbers in the body of the table indicate the number of times the situation at the left exists for all examples involving the divisor shown at the top.

This report does not include data for all two-digit divisors ending in 4, 5, or 6. Such an analysis could be made but a complete analysis may not be necessary in order to provide teachers with a basis for decision about methods of estimating trial quotients. In order to provide some basis for making inferences about the facts when other divisors are involved, complete data are provided in Table II for divisors 34, 35, and 36, and for divisors 84, 85, and 86.

EDITOR'S NOTE. Long division is generally considered the most difficult topic in the ele-

[2] Upton, C. B., *op. cit.*, p. 287.

TABLE I

SHOWING NUMBER OF TIMES VARIOUS TYPES OF EXAMPLES OCCUR WITH CERTAIN SELECTED DIVISORS

Types of Examples	Sample	Divisors								
		15	25	35	45	55	65	75	85	95
Dividend in same decade as divisor, quotient obvious, Rule B not applicable	$34\overline{)376}$	5	5	5	5	5	5	5	5	5
Both rules yield same trial quotient	$84\overline{)462}$	0	10	30	60	100	150	210	280	360
Difference between two trial quotients is 1, Rule A yields true quotient	$36\overline{)147}$	0	15	40	75	120	160	190	210	220
Difference between two trial quotients is 1, Rule B yields true quotient	$55\overline{)262}$	10	35	70	115	160	190	210	220	220
Difference between two trial quotients is 2, Rule A yields true quotient	$34\overline{)238}$	0	0	0	0	0	0	0	0	0
Difference between two trial quotients is 2, Rule B yields true quotient	$36\overline{)213}$	5	5	5	0	0	0	0	0	0
Difference between two trial quotients is 2, true quotient is number between	$36\overline{)261}$	15	55	105	100	60	30	10	0	0
Difference between two trial quotients is 3 or more, true quotient is between	$34\overline{)273}$	50	50	10	0	0	0	0	0	0
Rule A yields trial quotient of 10, Rule B yields true quotient	$86\overline{)831}$	0	0	0	5	10	20	30	40	50
Rule A yields trial quotient of 10, Rule B does not yield true quotient	$55\overline{)526}$	50	50	50	45	40	30	20	10	0

TABLE II

SHOWING NUMBER OF TIMES VARIOUS TYPES OF EXAMPLES OCCUR WITH CONSECUTIVE DIVISORS

Types of Examples	Divisors					
	34	35	36	84	85	86
Dividend in same decade as divisor, quotient obvious, Rule B not applicable	6	5	4	6	5	4
Both rules yield same trial quotient	30	30	30	280	280	312
Difference between two trial quotients is 1, Rule A yields true quotient	50	40	26	254	210	134
Difference between two trial quotients is 1, Rule B yields true quotient	60	70	84	176	220	264
Difference between two trial quotients is 2, Rule A yields true quotient	2	0	0	0	0	0
Difference between two trial quotients is 2, Rule B yields true quotient	0	5	26	0	0	0
Difference between two trial quotients is 2, true quotient is number between	108	105	84	0	0	0
Difference between two trial quotients is 3 or more, true quotient is between	10	10	10	0	0	0
Rule A yields trial quotient of 10, Rule B yields true quotient	0	0	4	30	40	50
Rule A yields trial quotient of 10, Rule B does not yield true quotient	40	50	56	10	10	10

mentary school. Professor Benz has given us a good exposition and sets of data showing how the application of *Rule A* and *Rule B* to certain cases does or does not produce the correct quotient figure. To be really intelligent about the division process and the relationships of the numbers involved requires a high order of thinking. It is not easy. Many of the devices including both *Rule A* and *Rule B* serve to make the process more mechanical and to lower the level of mental process. Other devices such as making a table of multiples of the divisor have been suggested. Mr. Benz presents a problem and some data and leaves the next steps to us as teachers and research workers. What shall we do? First of all shall we continue to expect all pupils to learn long division even though this is very painful to certain pupils? Shall we use a very different approach to division such as the old "scratch method" or one of the other methods of subtracting obvious multiples of the divisor? Shall we teach our more able pupils to think and estimate with the divisor-dividend relationship as a unit and not resort to special devices? What will we do with the less able pupils who find thinking very arduous? Who will tell us how best to teach long division, or whether there is a *one* method or if there are several methods we should be using?

Another Carnegie Report on Mathematics?*

Exertion to bring teachers to a higher standard will be more effective in improving school education than any efforts at improving school *books* can possibly be. It is here where the great improvement must be sought. Without the cooperation of competent teachers, the greatest excellences in any book will remain unnoticed and unimproved.

Then let teachers make themselves, in the first place, thoroughly acquainted with arithmetic. The idea that they can "study and keep ahead of their classes," is an absurd one. They must have surveyed the whole field in order to conduct inquirers over any part, or there will be liability to ruinous misdirection. Young teachers are little aware of their deficiencies in knowledge, and still less aware of the injurious effects which these deficiencies exert upon pupils, who are often disgusted with school education, because they are made to see in it so little that is meaning.

In the next place, let no previous familiarity with the subject excuse teachers from carefully preparing each lesson before meeting their classes. Thereby alone will they feel that freshness of interest which will awaken a kindred interest among their pupils; and if on any occasion they are compelled to omit such preparation, they will discover a declining interest with their classes. Teachers who are obliged to have their books open, and watch the page while their classes recite, are unfit for their work.

* From suggestions to teachers written by W. B. Bunnell in *Adam's New Arithmetic—Revised Edition*. Keene, N. H., J. H. Spalter & Co., 1848.

This item was sent in by Isaac Feinberg of Willoughby Jr. H. S., Brooklyn, N. Y. Does it sound familiar even though 100 years old?

Definitions in Arithmetic

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TEACHERS OF MATHEMATICS know how difficult it is to teach students to use definitions skillfully. Perhaps one explanation of the difficulty which even the better students seem to have in using definitions properly lies in the fact that in early arithmetic experiences they did not often work consciously with definitions. The writer feels that there are many rich opportunities in the early arithmetic years to allow youngsters to share in the construction of definitions. Moreover, in many arithmetic situations opportunities for critical examination of definitions arise naturally.

The definitions of arithmetic differ somewhat from many of the definitions of advanced mathematics. It is, of course, quite correct to say that every definition is simply an agreement to substitute one group of symbols or words for another. But in advanced mathematics many definitions are highly creative in nature, calling attention to new abstract concepts which might otherwise never be noticed by the student. On the other hand the formal definitions of arithmetic seldom call attention to new concepts. Rather, they pin new names upon old familiar concepts. In order to illustrate this, consider the introduction of the terminology of *addition.* Long before the teacher says, "Add four and three," the children have been solving addition problems. But they called it *counting.* And when the teacher first uses the word *add* a definition must be made. Perhaps the teacher simply says one day, "When I ask you to add four and three I am only asking you to count how many things there would be altogether if there were four in one group and three in a second." One casual remark like this may well serve to give a precise understanding of the definition of addition to an entire class. Here it is obvious that the definition entails no new mathematical ideas but only attaches a new word to familiar concepts.

Perhaps the most interesting concept that arises in the teaching of addition is the concept of the null or empty set. It does not seem difficult to get youngsters to accept the idea of a group which contains no members. When this is done, the zero combinations are easily grasped and a youngster might observe that $3+0$ can be interpreted as finding the number of things in *two* groups if there are three in one and zero in the other. At some stage in the addition process the teacher might present as a description of addition that it is *counting the number of things in two or more groups.*

The other definitions of arithmetic may be almost entirely erected upon that of addition. Perhaps we should remark here that a mathematician would observe that if this is done then surely arithmetic is built upon sand. For the mathematician would point out that actually addition *has not been defined by our remarks above, rather it has only been described.* This is certainly correct. And for the sake of those who would really like to see a precise definition of addition, we list one below where x and y represent arbitrary counting numbers and the symbol x's represents the number which comes after x in the counting process.

(1) $$x+1=x'$$

(2) $$x+y'=(x+y)'.$$

Equations (1) and (2) above can be used to prove the mysterious properties of addition such as $x+y=y+x$ and $(x+y)+z=x+(y+z)$.

Since this rigorous procedure does not seem advisable in the elementary school we proceed along more intuitive paths. There are two alternative definitions of subtraction. The one presents it as a taking away process, the other as the inverse of addition. Again, of course, children are excellent subtractors long before they hear the term *subtract*. They have solved many problems like: "Johnny ate two of his five jelly beans. How many has he left?" before the day the teacher says, "Subtract two from five." Again the only thing that needs to be done is to attach the new terminology to the already well understood ideas. It would be an interesting experiment to present only the one definition, namely, that "subtracting two from five" means "to find the number which must be added to two in order to get five."

At this point an opportunity for teaching critical thinking occurs. After some time has been spent on subtraction the teacher should say, "subtract eight from three." The careless student will probably give five as the answer. But the thinking student will say that *the problem can not be done*. Here is an opportunity to begin to teach the precision of mathematics.

Interesting vistas open up as we proceed. The child should know much about multiplication before it is formally defined. The definition will probably run something like the following: 4×3 means to add four threes. Now we ask the child, "What is one times three"? Let the reader at this point go back and reread the definitions of addition and multiplication and and decide how he himself, as a child, would have replied. The thoughtful child will observe that: "1×3 *has not been defined for it makes no sense to speak of adding one three. In order to add we must have two or more numbers.*"

This is of course correct. In general teachers are probably not conscious of the limitations of the definitions they make. Definitions must often be expanded. The same difficulty arises with an example such as 0×4. One way out of the difficulty is to *define* $1\times3=3\times1$ and $0\times4=4\times0$. But this still leaves the three products involving only the numbers 1 and 0 undefined. Special definitions must be made for them. Let us emphasize that the precise definition of multiplication is indeed a tricky business. It requires much time and an understanding teacher. At any stage in mathematics when a student does not react as the teacher thinks he should to a question, it might be well to go back and recall the preceding definitions which have supposedly prepared the student for the question. *Perhaps the definitions do not provide the student with what he needs to draw the conclusion we should like to have him infer.* It is well to recognize the fact that statements like "one times three" have no meaning until we give them meaning by definitions.

Let us suppose that we are over the hurdle posed by multiplication and consider the opportunities that present themselves in division. We may define "divide 12 by 3" to mean:

a. Separate a group of twelve things into three equal groups and count the number in one of these groups (Partitioning).

b. How many threes must be added together to make twelve? (The "Contained in" aspect).

c. How many threes may be subtracted from twelve?

d. By what number must three be multiplied to make twelve?

Again I suggest that it would be an interesting experiment to present one and only one of these definitions of division to a class. If this were done, definition (b) seem the simplest to teach although (d) is tempting. Perhaps some of the difficulty youngsters have with division comes from not having a unique definition to fall back upon in their early division work.

Opportunities for questions which will sort out budding thinkers from dullards abound here. Perhaps the most dramatic is to ask something like, "What is $4\div0$?"

If definition (a) has been presented the question is quite tricky. Only the very finest students will remark that *the question makes no sense, for to speak of dividing four into zero equal groups is gibberish.* Others will probably react stupidly by giving zero as the result, or will make a mild reasoning effort and conjecture that $4 \div 0$ must mean *not to divide*, hence the result is *four*. If definitions (b), (c), or (d) have been presented, a larger number of students will recognize the absurdity of the question.

If definition (a) has been employed we raise the question "Has $5 \div 1$ been defined?" What about the meaning of such quotients as $0 \div 4$ when we relate them to the various definitions? Is it not clear that definition (d) applies neatly to this last question while (b) and (c) are decidedly fuzzy around the edges?

Encourage Thinking

The extension of the definitions of the four funamental operations to cover computations involving fractions is most interesting. We submit some conversations with a small child who had learned in his early number experiences to work consciously with definitions. The child had slightly better than average intelligence. These conversations occurred between the ages of six and seven. The child knew the meaning of fractions but had no knowledge of writing them. That is, the *sound* "three-fourths" was well understood as being associated with a certain amount of pie, but written symbols such as "$\frac{3}{4}$" had not been defined. We indicate the teacher by (T), the child by (C).

(T) How much is one-half and one-fourth?

(C) It's just one-half and one-fourth. It's not two of anything.

(T) Don't you know a single fraction that stands for a half and a fourth?

(C) Oh, yes. That is just three-fourths.

(T) When we put fractions together like this we call it adding. Add one-half and one-third.

(C) I believe it would be five-sixths.

(T) What makes you say that?

(C) Well, I just imagined how much one-half of a pie and one-third of a pie would make if you put them together, and it looked like five-sixth.

(T) Can you prove that it really is five-sixths—you might cut them up.

(C) Oh, I see, I can cut the half into three sixths and the third into two sixths. It *is* five sixths.

(T) What do you suppose we mean when we say "4 times $\frac{1}{2}$"?

(C) I suppose that, just as for whole numbers, it would mean to add four halves That would be 2.

(T) Right. Multiply $9 \times \frac{1}{3}$.

(C) Three.

(T) $4 \times \frac{2}{3}$?

(C) Eight thirds. That would be two and two-thirds.

(T) How about $\frac{1}{2} \times 4$?

(C) That doesn't mean anything. You can't add one-half four.

(T) Of course, we must make a new definition. When I say "one-half times four" I just mean for you to take "one-half *of* four."

(C) I see. The answer is two.

(T) $\frac{1}{2} \times \frac{2}{3}$?

(C) one-third.

(T) $\frac{1}{3} \times \frac{1}{2}$?

(C) One-sixth

(T) $\frac{1}{2} \times \frac{3}{4}$?

(C) One and one-half fourths.

(T) Can you give that answer differently.

(C) Yes. Three-eighths or six-sixteenths

(T) $\frac{2}{3} \times \frac{3}{5}$?

(C) One-third of three fifths would be one fifth, so two-thirds of three-fifths would be two-fifths.

(T) $\frac{2}{3} \times \frac{2}{5}$?

(C) Lets see one-third of one-fifth is one-fifteenth. So one-third of two-fifths is two-fifteenths. Then two-thirds of two-fifths is four-fifteenths.

(T) $\frac{3}{2} \times \frac{1}{3}$?

(C) *That's silly. You can't take three halves of anything.*

(T) Of course not. I just wanted to see if you were thinking.

(T) What do you suppose we mean when we say, "Divide two by one-half"?

(C) Well, $12 \div 3$ means to find how many threes there are in twelve. There are four halves in two. Is the answer four?

(T) That's right. We define division of fractions just as we defined division of whole numbers. What is $3 \div \frac{3}{4}$?

(C) There is one three-fourths in each whole, and the three pieces left over made another three-fourths. The answer is four.

(T) $1 \div \frac{2}{3}$?

(C) There is one two-thirds in one and then one-third left over.

(T) That's right. But sometimes we give the answer differently. What do you do when you divide 6 by 4?

(C) I say it is one and a half because there is one four and half of another four in six.

(T) In the same way tell me how many two-thirds there are in one.

(C) Well, the third left over is half of another two-thirds, so the answer is one and one-half.

(T) Right. And $1 \div \frac{3}{4}$?

(C) One and one-third.

(T) $\frac{1}{2} \div \frac{1}{3}$?

(C) Ill have to cut them both up into sixths. Let's see. The half makes three sixths and the third makes two. So I have one and one-half thirds in one-half.

"+" Has a Variety of Connotations

Several comments should be made. First and foremost, as we extend our opertions to fractions, new definitions of the fundamental operations are required. As much as possible we hang on to our old definitions. But it is clear that the plus signs occurring in $3+4$ and $\frac{1}{2}+\frac{1}{3}$ have far different meanings. We call each one "plus" but the first one says briefly "count," while the second says "cut into pieces of the same size and then count." This necessity of redefining addition occurs several times as the student progresses through high school mathematics. The student will be given new definitions for each of the following situations

$$(+8)+(-4)=0$$
$$(2x-y)+(-3x+4y)=0$$
$$\sqrt{2}+\sqrt{3}=0$$
$$(2+3i)-(1-4i)=0$$

In each case above the same familiar sign "+" is used, but in each case a different operation is meant. We call each of these operations "addition," but each has a different definition. The student who continues to study mathematics will learn to "add" such strange things as *vectors*, *quaternions*, *matrices*, and *point sets*. Always a new definition will be formulated.

The important thing to observe when multiplication of fractions is introduced is that the old definition for multiplication of integers breaks down completely. The definition that $\frac{1}{2} \times \frac{2}{5}$ means $\frac{1}{2}$ of $\frac{2}{5}$ is eminently satisfactory. In passing we note in sorrow that many teachers, particularly in percentage problems, offend common sense by telling their bewildered pupils that 15% of 200 means 15×200. Here is the clearest possible illustration of reversing the horse and cart.

If division of whole numbers is recalled as a partitioning process, i.e. $12 \div 4$ means to separate 12 into four equal groups etc., it is clearly apparent that this particular definition is not immediately extensible to fractions. For, if we consider $2 \div \frac{1}{4}$, what meaning is conveyed by the words, *separate two into one-fourth equal groups*?

The writer feels strongly that, in genneral, children are not taught arithmetic in such a manner that they have reasonable opportunity to grasp the definitions of the fundamental operations. After all, almost the entire aim of arithmetic instruction in the first seven grades is to develop a thorough understanding of the rational number system. To achieve this aim it seems necessary at every grade level for teachers to review the definition process and attempt to correct erroneous

impressions. One of the greatest weaknesses in our teaching is that we give youngsters algorithms to use as tools for solving problems when we should instead be helping them formulate clear definitions.

EDITOR'S NOTE. Dr. Brumfiel clearly establishes the idea that definitions are sensible agreements of what we mean and that they must grow as children encounter new areas in which the former definition no longer suffices. Most important is the idea that children can and should be conscious of the formation of definitions. But let us not try to be too adult in our statements, rather it is the elements of understanding and appreciation that are important. It is amazing how much mental work and what a high degree of logic children exhibit when they are encouraged to do so. Many teachers will find it hard to believe the answers of the youngster less than seven years of age. The work of this pupil illustrates beautifully a child's reasoning when meaning and understanding are fostered and the written algorisms are delayed. Isn't our aim in arithmetic primarily to get good thinking and good work leading to correct conclusions so that our young people and adults will be really intelligent in the many mathematical aspects of life in our complex society?

I Hated Arithmetic

Arithmetic in my childhood is a most unhappy memory. I readily memorized counting by one's, two's, five's and ten's as far as seemed limitless. The addition facts were as readily memorized. What a bright child! Then came subtraction. That was not so easy. I found there were others who excelled me. Simple problems, measures and simple fractions were extremely difficult. I was beginning to hate arithmetic. Then once again, I memorized the multiplication table and some of my fame returned. Applying all this knowledge? No. The only way I could ever get my lesson was to have some one do it for me—not the teacher. I was ashamed to "let her know." I copied from others. Often we worked at the board. I could do nothing unless I copied. Always I was in tears at the close of the period. I hated arithmetic.

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Why My Children Like Arithmetic

I am interested in guiding my children to do mathematical thinking by giving them opportunities to participate in learning situations. Because the child's arithmetic learning depends much on the teacher and the instructional material and procedures she uses, I think my children's happy attitude toward arithmetic grows from my effort to develop the "concept of understanding."

When we go back to school this fall the slow learner will have an opportunity to participate. Some arithmetic topics will be presented to the class as a whole. The immature children may take part with the simple details while the ones with a higher level of understanding will develop the more difficult phases of the lesson.

The children (each child working on his own level of ability) will count by one-to-one correspondence in sequence from one to ten, find the relationship of the symbol and the quantity it represents, count by groups and recognize the number in the group without counting by ones. They will discover relationships of numbers. By adding one to a number they will find the next number. By subtracting one from a number they will find the number that comes just before. They will compare groups of numbers, find differences and find what is left when part of the group of numbers has been used.

The children will have fun making change with money by adding on from the price of the article to the amount of money possessed in the beginning. They will learn to find a fractional part of their cup cakes, and divide their candy among the children of a small group. These experiences with real things and with numbers will teach much arithmetic on the first grade level and will also lay a basis of understanding things that come in higher grades.

Reported by BERNICE COOPER
First Grade
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The Ten-Tens Counting Frame

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IN THE NOVEMBER 1955 issue of the ARITHMETIC TEACHER there appeared and article entitled *The Abacus as an Arithmetic Teaching Device.* It dealt mainly with computation. This article does not compete with it for this has to do with the abacus (we call it a Ten-Tens Counting Frame) not as an instrument for computation but as a means of developing an understanding of our number system.

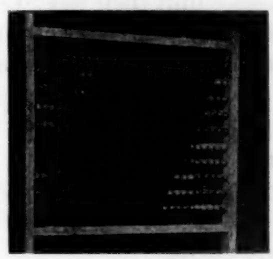
We have a remarkably simple number system based upon *ten* and *place value.*, Children have to learn the symbols 1, 2 3, 4, 5, 6, 7, 8, 9, and 0. All other numbers are written as combinations of these digits. Think of the simplicity of our system as compared to the Roman system. We can write any number, no matter how large using only the digits mentioned. In the Roman system new symbols had to be learned, sometimes invented, when it became necessary to write large numbers. Did you ever try to multiply a number like MCMIX by XXIV?

Childrens' understanding of the number system is one of the unifying concepts that makes arithmetic more meaningful and easier to grasp. As a trivial example, a child who understand that 34 means 3 tens and 4 ones is not so likely as others to write 43 when asked to write thirty-four. For another example, the glibly given statement, "Put down the 2 and carry the 1" can be explained to a child who knows that 12 is 1 ten and 2 ones.

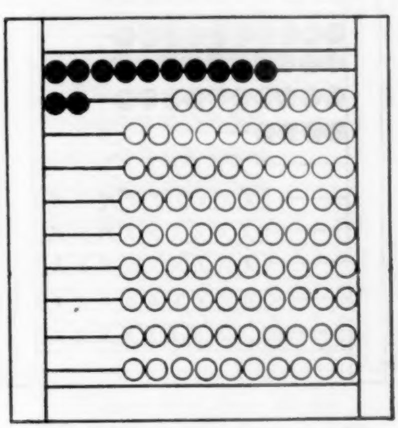
Following are suggestions for using the Counting Frame in developing an understanding of our number system. It is not intended that this material be taught as a unit in any grade but spread out through the early years of the elementary school as the children are ready for it.

Children should have many experiences with integers from 1 through 20 before they begin a study of the number system. The Counting Frame is an instrument for helping children with the number *system* not with the elementary ideas of number or counting with numbers.

The Counting Frame as we use it in Springfield is a large one, large enough to be used as a demonstration instrument set up in the front of the room. It consists of ten rows of ten beads each. For convenience the beads are colored by fives.



The numbers from 11 through 20. The number 11 is 1 ten (1 row across) and 1 bead more (1 bead on the next row.) The number 12 is 1 ten and 2 ones. And so on. These ideas can be shown very concretely on the Counting Frame.

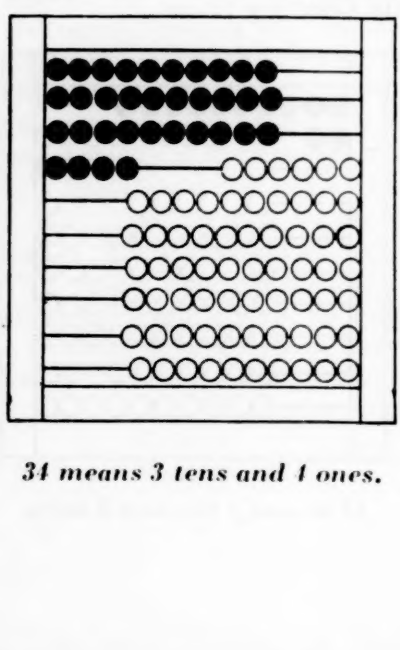


12 means 1 ten and 2 ones.

The decades, ten, twenty, thirty, . . . As the names twenty, thirty, forty, etc. are given the teacher can move rows on the Counting Frame giving a concrete representation of the numbers. (Did you ever stop to think how much easier our teaching in the first grade would be if we could say, "onety," "twoty," "threety," "fourty" and "fivety" as well as sixty and seventy?) Children should practice with these new words and the numbers they indicate while one of them moves the corresponding beads.

Counting beyond twenty. Counting to one hundred by ones should now be easy. The teacher can move three rows as the children say, ten, twenty, thirty. Then she can move one bead on the next row. The children will invariably say forty, but they soon see and admit their mistake. This, of course, is thirty and one, or thirty one. Move another bead. You then have thirty and two, or thirty-two. Since children already know how to count from one to ten, they should have no difficulty with the thirtys and higher. This procedure can be repeated with any of the decades until the children are adept.

Meaning of a two-figure number. Ask a child to move thirty-four beads. He may start counting by ones. He should move 3 tens and 4 ones. Ask the class to write 34 copying what you have written on the



34 means 3 tens and 4 ones.

board. This is *thirty-four*. The 3 represents 3 tens as they have seen on the Counting Frame. The 4 represents 4 ones.

This same process may be carried through with many other two-figure numbers. For example, let the children see the difference in meaning between 34 and 43. One means 3 tens and 4 ones, the other 4 tens and 3 ones.

Column addition. It often happens that numbers such as 6, 7, 5, and 6 have to be added before children are ready to do it abstractly. The addition can be done on the Counting Frame. Move 6 beads. Then there are 4 beads left on the first row. Ask how you can move 7 beads when there are only 4 left. The answer is, "Move those 4 and 3 on the next row." Then move 5 beads and then 6 beads. The answer can be seen at a glance. It is 2 tens and 4 ones, or 24.

This example is typical of many examples that can be done on the Counting Frame even before the children are ready to study the abstract aspects of them.

Bridging the tens. Ask an adult why the sum of 8 and 5 is ordinarily given as thirteen and not as one dozen and one. He probably will not know. The reason is that we have a *ten* system and not a *twelve* system. In a ten system the sum of 8 and 5 is written 13 (one ten and 3 ones); in a twelve system it would be written 11, one twelve and 1 one). At the beginning we might ask, "8 and 5 are ten and how many?" The Counting Frame will help with the answer.

To find 8 and 5, move 8 beads. How many beads are left on the row? The answer is two. If we move these 2 beads and wish to move five beads, how many more beads do we need to move. Here the answer is 3. The sum of 8 and 5 is seen to be 1 ten and 3 ones or 13. Practice with many combinations, 9 and 2, 9 and 3, 8 and 4, 9 and 7, and many more.

The same procedure can be used for higher decade addition. 18 and 5 is 3 on the next row. 28 and 5 is 3 on the next row.

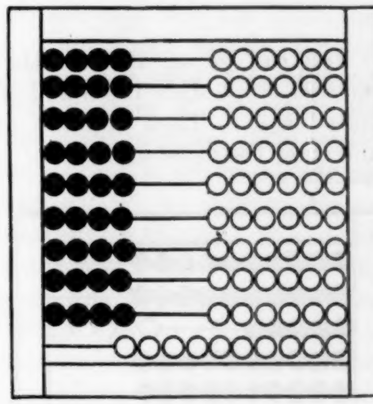
38 and 5 is 3 on the next row. Of course it is necessary to know what the next row is. In the first case it is twenty, then thirty, then forty. The answers are 23, 33, and 43.

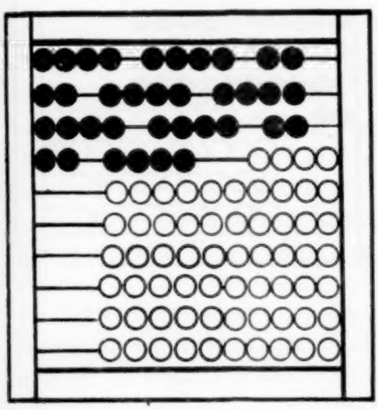
Bridging the tens backward. This same idea can be used in reverse. If you wish to subtract 7 from 13, set up 1 ten and 3 ones. Take away the 3 ones. Then you have 4 more to take away. The answer is the same as 4 from 10 or 6.

Adding two-figure numbers without carrying. You wish to teach the addition, say, of 23 and 35. You wish to show that the sum is found by adding the tens and the ones separately. Set these numbers up on the Counting Frame, 2 tens and 3 ones and then 3 tens and 5 ones. Ask a pupil to see how many there are in all. If he begins to count by ones suggest that there is a better way. He should count the tens alone and the ones alone. There are 5 tens or 50. There are 8 ones or 8. There are 58 beads in all. Whether the tens or the ones are counted first, makes no difference.

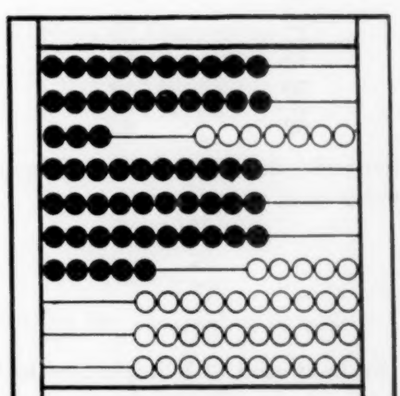
Multiplication combinations. Take for example, nine 4's. One way is to take nine 4's as shown and ten count the beads. A better way is to move them as shown in the second diagram. Then when the beads are pushed together, the answer can be seen at a glance, 3 tens and 6 ones, or 36.

FIRST METHOD
Nine 4's are 36.





SECOND METHOD
Nine 4's are 36.



Count the ones alone then count the tens alone

At first

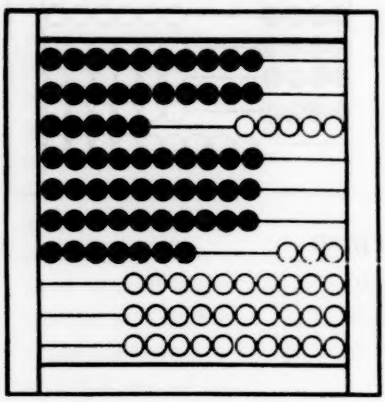
$$\begin{array}{r} 23 \\ +35 \\ \hline 8 \\ 50 \\ \hline 58 \end{array}$$

Finally

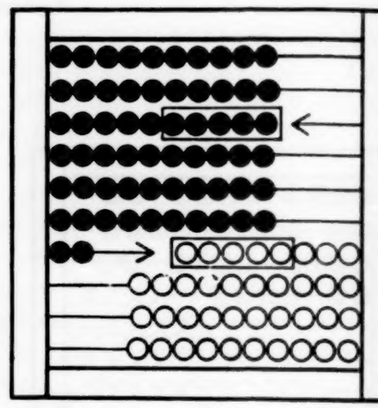
$$\begin{array}{r} 23 \\ 35 \\ \hline 58 \end{array}$$

You can go on then to the abstract method. First show the 8 and 50 as illustrated above. Then you can abbreviate and show the usual method.

Adding two two-figure numbers with carrying. You wish to find the sum of 25 and 37. Again, the tens and the ones should be added separately as can be shown on the Counting Frame. But after that there is more. The twelve ones must be changed to 1 ten and 2 ones. Count the tens alone. There are 5 tens or 50. Count the ones alone. There are 12 ones or 12. 50 and 12 are 62. There are sixty-two beads in all. To make the counting easier change the 12 ones to 1 ten and 2 ones as illustrated in the diagram. Then there are 6 tens and 2 ones, 62 as before.



The 7 and 5 of the previous diagram have been changed to 1 ten and 2 ones.



Now go on to the algorism by steps as shown below.

First

$$\begin{array}{r} 25 \\ +37 \\ \hline 12 \\ 50 \\ \hline 62 \end{array} \quad \text{or} \quad \begin{array}{r} 25 \\ +37 \\ \hline 50 \\ 12 \\ \hline 62 \end{array}$$

Then write the 12 as shown, 12 being 1 ten and 2 ones.

$$\begin{array}{r} 1 \\ 25 \\ 37 \\ \hline 62 \end{array}$$

As we have said, it is important in the development of arithmetic for the pupil to understand the number system. It has been said that if children could understand the system as well as mathematicians do, the teaching of arithmetic might be eliminated. We can never hope for that but we can make sure that the number system is brought to our young people in such a concrete way that it is meaningful. The Counting Frame is a major means to that end.

EDITOR'S NOTE. Dr. Smith, as coordinator of mathematics for the Springfield Public Schools, has long been interested in helping teachers and pupils understand arithmetic. He has pointed out ways of using the abacus for understanding the number system. He shows how using it in simple computations enhances understanding when a teacher makes suitable explanations. There are many possible similar devices which range from clothes pins on coat hangers or tongue depressors in pockets to well constructed frames with beads or counters. With all of these devices, it is the *time of use* and the *way in which they are used* which give them significance in aiding learning. Teachers who have never used a device such as Dr. Smith's abacus might well experiment, first with themselves, and then with their pupils to see how they can improve their pupils' learning.

Decimal Versus Common Fractions

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Common fractions were being used in Egypt at about 1600 B. C. and Decimal fractions were invented about 1600 A. D. Thus the common fraction has had a 30,000 year handicap over the decimal by virtue of its priority in time. This handicap has been so strong that its influence is still in force as evidenced by the fact that the common fraction is taught before the decimal despite the evidence that decimals are easier to learn and to use.

Shortly after Simon Stevin had developed and described the decimal fraction in 1585 an arithmetic text emphasizing the simplicity and the advantages of the decimal and almost excluding the common fraction was printed. Later, several authors did the same in this country. But the habit of teaching and using common fractions for so long a time had produced

01079		45				
31080	0	46	35	47	0	53
206	50	0	37			
30873	10	45	58	47	0	53

a technique and routine that was hard to discontinue and break. Teachers today have been taught common fractions before decimals and the textbooks of today dare not break away from the tradition of placing common fractions first because publishers fear they would not sell and so we continue in the old rut.

Away back in the middle ages the sexagesimal fraction was used in which 60 was the ratio in place value. An example from a 14th century text book in arithmetic will be shown here so that the reader can study out how to subtract in the example. A translation from the Hebrew is given here to help the reader see how it was done.

"Then we subtract from the 35 in the upper row that which stands directly under it in the lower, that is 37, that we cannot subtract from 35, therefore we take one out of the heading which comes after 35, in this rank it will be 60, adding it to 35 makes it 95, we subtract therefrom 37, thus leaving 58 which we write in the row of the result under the heading, Terzen; hence leaving only 45 left in the following order"

The above method was used as late as 1640. We wonder now why people suffered so long with such a clumsy method. Since a two-figured number was used in each sexagesimal fraction a space had to be provided between the different places. Perhaps in the not far distant future people will wonder in a similar vein when looking at an example in subtraction such as this

$$\begin{array}{r} 245\frac{1}{5} \\ 167\frac{3}{4} \\ \hline \end{array}$$

which has to be changed to

$$\begin{array}{r} 245\frac{4}{20} \\ 167\frac{15}{20} \\ \hline \end{array}$$

and then to

$$\begin{array}{r} 244\frac{24}{20} \\ 167\frac{15}{20} \\ \hline \end{array}$$

while they are comparing it with

$$\begin{array}{r} 245.2 \\ 167.75 \\ \hline \end{array}$$

where they can subtract right off just as they do with whole numbers.

That is to say we have to have our children learn all of the four operations in addition, subtraction, multiplication and division not only in whole numbers where place value is the central theme of understanding but in common fractions where there is no place value causing the operations to be performed in an entirely different manner subject to rules altogether different when there is a simpler way that operates by the principle of place value thus using and reenforcing a technique which is already known.

That decimals are preferred by the outside world is evidenced by the fact that mechanics has decimalized the inch. The auto machine shop uses the .0001 of an inch in its measurements. Rainfall and atmospheric pressure are quoted in inches and tenths. Engineering has decimalized the foot. The surveyor's tape is graded in feet and tenths not feet and inches. Aviation has decimalized the mile. We read of the jet bomber making a flight at 686.725 miles per hour not miles and common fractions of a mile.

I am told that men in the machine shop when they have common fractions to add or subtract use the decimal equivalents while performing the operation and then transcribe the results back to common fractions.

I want to tell briefly of an experiment on this question conducted some years ago while teaching at Chicago Teachers College. Permission for the experiment was obtained from the Principal and 5th grade teacher of the nearby Harvard School. In early March of that year a 5B class was taught the decimal equivalents to the 15 most frequently used common fractions given below with their equivalents in four groups in order of difficulty.

$\frac{1}{2}=.50$	$\frac{1}{5}=.20$	$\frac{1}{8}=.125$	$\frac{1}{3}=.333$
$\frac{1}{4}=.25$	$\frac{2}{5}=.40$	$\frac{3}{8}=.375$	$\frac{2}{3}=.667$
$\frac{3}{4}=.75$	$\frac{3}{5}=.60$	$\frac{5}{8}=.625$	$\frac{1}{6}=.167$
	$\frac{4}{5}=.80$	$\frac{7}{8}=.875$	$\frac{5}{6}=.833$

After fourteen weeks of learning the above and practice in using them in short examples of addition and subtraction of common fractions the following test was given to the 5B group and also to a 5A class as a control group. The 5A control group worked these examples as they had been taught using common denominators while the 5B experimental group changed the common fraction to its decimal equivalent first and then added or subtracted.

Addition and Subtraction of Common Fractions

Add the first four examples. (Use the space at the right of each example to work in)

(1)	$3\frac{1}{2}$ $4\frac{1}{4}$	(2)	$7\frac{1}{3}$ $1\frac{7}{8}$
(3)	$5\frac{1}{10}$ $6\frac{3}{4}$	(4)	$2\frac{3}{4}$ $\frac{5}{6}$

Subtract the following six examples. (Use the space at right of each for working)

(5)	$4\frac{3}{4}$ $1\frac{1}{8}$	(6)	$12\frac{2}{3}$ 9
(7)	10 $5\frac{3}{8}$	(8)	$9\frac{2}{3}$ $6\frac{3}{4}$
(9)	$6\frac{2}{3}$ $\frac{7}{8}$	(10)	$\frac{5}{6}$ $\frac{3}{4}$

The large difference in the results in favor of the 5B group in both time and accuracy as shown above gave the writer the desire to repeat the experiment the next year with a still wider difference in mental ages. The conditions under which the tests and practice were done were the same as in the first experiment with the difference that the control group was chosen from the best 6A class in the school, three semesters above that of the experimental group.

Results of the First Experiment

Grade	No.	Accuracy, Percent			Time, Minutes			Mental Age, Years			I.Q.		
		Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$
5B	46	94.3	9.2	1.38	6.8	1.6	0.24	11.7	0.91	0.13	106.0	12.6	1.87
5A	37	75.3	14.8	2.37	19.6	5.7	0.92	12.1	0.91	0.14	106.1	10.7	1.73
Differences		19 percent			12.9 minutes			0.4 years			0.1		

Results of the Second Experiment

Grade	No.	Accuracy, Percent			Time, Minutes			Mental Age, Years			I.Q.		
		Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$	Mean	S.D.	$P.E._M$
5B	44	96.1	10.0	1.5	11.2	2.4	0.37	11.4	0.9	0.14	106.6	12.0	1.81
6A	40	88.5	11.6	1.8	14.7	3.3	0.52	12.8	1.1	0.18	106.5	11.4	1.81
Differences		7.6 percent			3.5 minutes			1.4 years			0.1		

The above results show without a doubt how much easier it is to learn and use the decimals in addition and subtraction of fractions.

Three Important Reasons Why Decimals Could Be Taught First

First, after United States money is taught in the fourth grade it is an easy step to one- and two-place decimals instead of breaking in with common fractions with their entirely different kinds of operations. There is no place value in common fractions, hence an entirely new system of procedure has to be taught throughout addition, subtraction, multiplication and division of common fractions and before any of these four fundamental operations in whole numbers are mastered.

Second, a needed continuation and reenforcement of the place value concept takes place in decimals after operations in whole numbers have been only partly learned.

Third, it is easier to find and correct errors for both teacher and pupil. In addition and subtraction of decimals there are only two new and comparatively easy skills to learn, viz., the writing of decimals and the keeping of the decimal point in a column. Whereas in common fractions there are at least eight separate and distinct rather difficult skills to teach and learn in connection with the different combination of denominators and mixed numbers.

It is not the idea of the writer that the common fraction should be thrown out. The $\frac{1}{2}$'s, $\frac{1}{4}$'s and eighths should be learned better than they are so that they can be used without a pencil. But since adult usage favors decimals, teachers should be willing to emphasize decimals more than common fractions as percentage which we all should know is based on the decimal. If teaching the decimal before the common fraction seems a bit too drastic they can both be taught and introduced at the same time in the fifth grade and let the pupil decide which he can learn and use the better. This has been done with surprisingly good results.

EDITOR'S NOTE. Dr. Johnson attributes the earlier place of common fractions over decimal fractions in textbooks and the curriculum to tradition and historical circumstance. This may not be the case. The common fraction concept is basically one of natural division or fracturing of a whole thing or a group, first into perhaps two parts, and later into more complex sections.

[*Editor's Note continued on page 206*]

More About Casting Out Nines

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CONSIDERING THE RECENT INTEREST in proof by nines, one might conclude this is a mid-twentieth century discovery. Probably we can concede it is a rediscovery. In *University Arithmetic* by Charles Davies, LL.D., A. S. Barnes & Co., New York, 1868, the process is explained and is used throughout the book. In *Advanced Arithmetic*, Teachers Edition, G. A. Wentworth, Ginn & Co., Boston, 1898, the check by nines is called for and no explanation given, indicating it was expected to be part of a teacher's store of arithmetic knowledge.

Even fifth graders can learn to use the check, and higher grade pupils usually find it fascinating. Unless care is taken, it becomes just another mechanical device, as does the caret in division or the carried number in addition. We lose sight of the mathematics involved and the mathematics is what makes the process interesting. But teachers who criticize proof by nines as being a mechanical, meaningless procedure are letting their ignorance of our number system show.

To understand why casting out nines gives a fairly accurate, easy form for checking the arithmetic operations, we must consider the nature of out number system. Our number system is a decimal number system, base 10. Each order has a value ten times the value of the next smaller order, or a value 10^{n-1}. Second, in the order of our number names, ten is one more than nine.

Ten to the first order is 10,
that is, $9+1$.
Ten to the second order is 100,
that is, $99+1$.
Ten to the third order is 1000,
that is, $999+1$.

Each order of ten is one more than some multiple of nine. Thus:

$$\begin{aligned}743 &= 700+40+3\\ &\ 7(100)+4(10)+3(1)\\ &\ 7(99+1)+4(9+1)+3(1)\\ &\ 7(99)+7(1)+4(9)+4(1)+3(1)\end{aligned}$$

Now, when the multiples of nine, called nines, are removed, there is left $7(1)+4(1)+3(1)$, or $14(1)$, fourteen ones. This equals the sum of the digits of the number 743. When 9 is removed from the 14, there is left 5, which is the residual of 743 divided by 9. So, the residual of a number divided by nine is the sum of the digits of the number. The residual may, in turn, be simplified.

Applying this rule to an addition problem—using M to represent the multiple of nine; E, the excess of residual (1, 2, 3, etc.) of the respective addends; and S, the sum we have:

$$\begin{array}{r}3842 = M_1+E_1\\ 5634 = M_2+E_2\\ \underline{7159 = M_3+E_3}\\ 10635 = M_1+M_2+M_3+E_1+E_2+E_3\end{array}$$

	residuals	
3842	8	
4634	0	
7159	4	
10635	12	3

$$M_s+E_s = M_1+M_2+M_3+E_1+E_2+E_3$$
$$E_s = E_1+E_2+E_3$$

Remove the M's.

The excess of the sum is equivalent to the sum of the excesses of the addends.

Using the subscripts 1 and 2 to represent the minuend and the subtrahend and d to represent the difference, we subtract:

$$\begin{array}{rl} 8435 & M_1+E_1 \\ -4368 & M_2+E_2 \\ \hline 3797 & M_1-M_2+E_1-E_2 \end{array}$$

residuals

$$\begin{array}{rr} 8435 \ldots\ldots\ldots\ldots\ldots & 2 \\ -4368 \ldots\ldots\ldots\ldots\ldots & 3 \\ \hline 3797 \ldots\ldots\ldots\ldots\ldots & 8 \end{array}$$

$$8+3=11=2$$

Remove M's.

$$M_d+E_d=M_1-M_2+E_1=E_2$$

$$E_d=E_1-E_2$$

$$\text{or,}\quad E_d+E_2=E_1$$

For subtraction, the excess of the minuend less the excess of the subtrahend equals the excess of the difference. Or, applying the axiom of equal additions, the excess of the minuend equals the sum of the excesses of the difference and the subtrahend.

In multiplication, the excess in the product equals the product of the excesses in the factors.

Residuals

$$\begin{array}{rrr} 743 \ldots\ldots\ldots\ldots\ldots & 5 & \\ \times 305 \ldots\ldots\ldots\ldots\ldots & \times 8 & \\ \hline 3715 & 40 & \ldots\ldots 4 \\ 2229 & & \\ \hline 226615 \ldots\ldots 4 & & \end{array}$$

In dividing, the product of the excesses in the divisor and the quotient plus the excess in the remainder equals the excess in the dividend.

In checking by casting out nines, the same rules are used as in ordinary operations, but the proof is by using the smaller residuals instead of the larger original numbers.

Applying the order series expressed with exponents, 743 may be written

$$7(10)^2+4(10)^1+3(10)^\circ.$$

Substituting the $(9+1)$ for 10, this becomes

$$7(9+1)^2+4(9+1)^1+3(9+1)^\circ.$$

Expanding gives:

$$7[(9)^2+2(9)(1)+(1)^2] \\ +4(9+1)^1+3(9+1)^\circ$$

$$7(9)^2+7(2)(9)(1)+7(1)^2 \\ +4(9)+4(1)+3(1)^*$$

Removing multiples of nine leaves

$$7(1)^2+4(1)+3(1)$$

Again we have 14(1) or 14, the sum of the digits of the number 743.

Since we have come this far in rationalizing the theory of casting out nines, let us go a step farther. Instead of the base 10, let us use a general number in base X, and divided by one less than the base.

$$\begin{array}{r|l} & 7X+11 \\ \hline X-1 & 7X^2+\ 4X+\ 3 \\ & 7X^2-\ 7X \\ \hline & \quad 11X+\ 3 \\ & \quad 11X-11 \\ \hline & \quad 14, \text{ remainder} \end{array}$$

The remainder 14 is the sum of the coefficients of the orders of original expression. This is just repeating the rule: When a polynomial in any base is divided by the base less one, the constant remainder is the sum of the numerical coefficients in the polynomial divided.

So, if

$$A(10)^n+B(10)^{n-1}+C(10)^{n-2}+\cdots \\ +Y(10)^1+Z(10)^\circ$$

is divided by $(10-1)$, the residual will be

$$A+B+C+\cdots Y+Z$$

The Remainder Theorem states: If the polynomial $f(x)$ is divided by $(x-r)$ until a constant remainder R is obtained. Then $R=f(r)$. That is, the remainder R is equal to the result when r is substituted

* Any quantity to the zero power has a value of one.

for x in $f(x)$. Our number 743 is a polynomial $f(10)$. If it is divided by 9, that is by $(10-1)$, until a constant remainder is obtained, the remainder will be equal to the result when 1 is substituted for 10.

Proof by nines lends itself to the checking of all the fundamental operations with integers.

MULTIPLICATION:

	excesses	
743	5	
×33	× 6	
2229	30	3
2229		
24519	3	

The excess of the product equals the product of the excesses of the factors.

DIVISION:

Excess	*Excesses*	
	37 1	
1 . . . . . . 46)	1732 4	
	138	
	352	$4=(1)0(1)+3$
	322	$4=1+3$
	30 3	

In division, the dividend equals the divisor times the quotient plus the remainder. So, the excess of the dividend equals the product of the excess of the divisor times the excess of the quotient plus the excess of the remainder.

Since the excess of nines in any number is the sum of the digits of the number, a rearrangement of the digits will not affect the excesses. Care must be taken to transcribe all numbers correctly. Check by casting out nines will not help detect an error made by transposing digits or an error when the sum of the digits is the same as the correct sum.

Proof by nines is not an infallible check, but it affords reasonable accuracy, it is interesting, and it is time saving.

It is worth adding to the arithmetic repertory. But do not accept it as another mechanical device. Too much arithmetic computation is mechanical. Probably because too much arithmetic teaching is mechanical. Arithmetic could be, and should be, one of the most ALIVE courses in the curriculum.

EDITOR'S NOTE. Teachers have expressed a wish for a mathematical proof of the validity of casting out nines. Miss Moore has given this and illustrated it in both number exercises and in general symbolism. She has also pointed out several of the circumstances under which the check fails to detect error. This is a check that is not purely mechanical. It has some good simple arithmetic involved and as Miss Moore says it is interesting and saving of time. A recent poll of teachers who had used the check showed that they were about equally divided in recommending its use to others.

Johnson—Decimal Fractions

(*continued from page 203*)

It is interesting to note that ancient civilizations seem to have developed a common fraction idea in their earlier stages and probably much later invented such fractions as minutes, decimals, etc. Certainly, the decimal fraction is much easier to use in computation and particularly so with modern machines. But, is the decimal easier to understand? Can we dispense with common fractions? Why did the early attempts to replace the common with the decimal fraction fail? Was it the schoolmasters of the time that wrecked the work of Erastus Root?

Some thirty years ago in a number of rural schools where all grades met within one room, the work of grades five and six was taught alternately and thus many youngsters learned decimal fractions before common fractions. We have no real records of the worth of these experiments. Perhaps now is the time to revive an interest in earlier teaching of decimal fractions and perhaps do as Dr. Johnson suggests, "Let the pupil decide which he can learn and use the better." What do school people think of Dr. Johnson's proposition?

A Plan for Teaching Arithmetic Shorthand

JEN JENKINS
Bethany College, Lindsborg, Kansas

How to Use the Chart

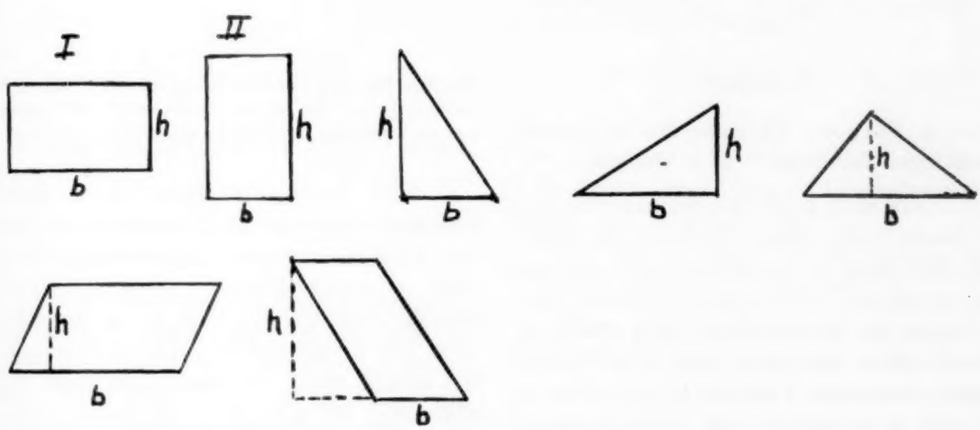
1. Clip a piece of blank cardboard over each of the sectional parts of the chart. Remove the cards, one at a time, as the material on the chart is needed—that is, after the pupils have had an explanation (from the teacher) and perhaps the experience of discovering some of the facts involved. After the topic has been taught, hang the chart in a conspicuous place in the classroom where the pupils may review the material at various times. There is much silent teaching done by such a display and therefore a wonderful opportunity would be lost if the teacher failed at this point.

2. Be sure to use the procedures suggested below in the order listed.

3. This presentation should not be made in one class period; rather, it should take a part of two or three consecutive class periods to give enough time for the concept to grow.

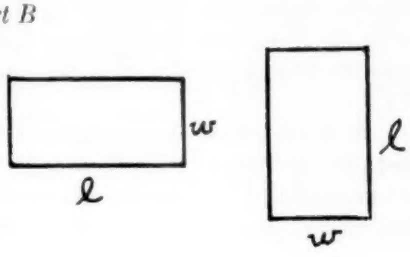
Chart of Arithmetic Shorthand

Part A



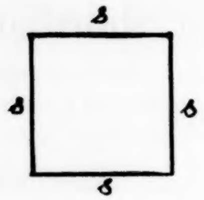
The base of a figure is the line on which the figure rests.
The height of a figure is the line which shows the shortest distance between the base and the highest point.
Arithmetic shorthand for "base" is "*b*."
Arithmetic shorthand for "height" is "*h*."

Part B



The length of a rectangle is the longer side.
The width of a rectangle is the shorter side.
Arithmetic shorthand for "length" is "*l*."
Arithmetic Shorthand for "width" is "*w*."

Part C

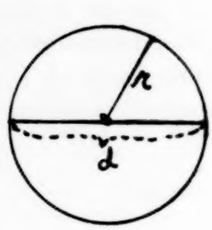


The side of a square is called "side."
Arithmetic shorthand for the side of a square is "*s*."

Part D

Arithmetic shorthand for "base times height" is "*bh*."
Arithmetic shorthand for "length times width" is "*lw*."
Arithmetic shorthand for "4 times side" is "4*s*."
Arithmetic shorthand for "*A* times *B*" is "*AB*."

Part E



Arithmetic shorthand for "circumference" is "*c*."
Arithmetic shorthand for "diameter" is "*d*."
Arithmetic shorthand for "radius" is "*r*."

Procedure

Part A, Purpose: To teach the arithmetic shorthand for "base" and "height."

1. Give each pupil two equal rectangles of construction paper. Pupils are to follow the procedure of the teacher who also has two rectangles of the same size and who, to make her presentation most effective, should place her rectangles on a flannel board. Rectangle I should be placed so as to rest on its longer side. The teacher's explanation should go something like this:

"The rectangle is resting on its longer side. We call the side it rests on its base, just as we call the part a lamp rests on its base. In order not to write any more than is needed, we use the initial of the word "base" just as people use initials for names. We write the letter "*b*" along the line. (Both teacher and pupils do this.) The other dimension is the height. We use its initial, "*h*," to represent it. We write the letter "*h*" along the short line of the rectangle. (Both pupils and teacher do this.) Notice that the height is the shortest line one could draw between the base and the highest point of the figure. Now let's place our second rectangle so that it rests on its shorter side. We can now call the shorter side the base, because the figure is resting on it. The longer side is now the height. Let us write the two letters where they belong."

2. Give each pupil three equal right triangles. Repeat the procedure of Section 1 for the triangles, first using the two sides as bases. Finally, place the right triangle such that it rests on its hypotenuse. The pupils will find that in this case, the height must be drawn through the triangle in order to show the shortest distance from the base to the highest point on the figure.

3. Give each pupil two equal parallelograms which have no right angles. Using each side as a base, in its turn, the pupils will find two lines to use as bases and two lines to use as heights. The latter pair will have to be drawn in the figures. Extra practice in this procedure may be necessary.

4, 5, 6. Each of the manipulations described in sections 1, 2, 3 should be shown graphically on both the blackboard and on paper. (See Part A on the chart.)

Beside the figures should be written statements which define "base" and "height" and which show the shorthand for the terms. (See Part A on the chart.)

7. Pupils should now see the chart, Part A, and study and discuss it.

Part B, Purpose: To teach the arithmetic shorthand for "length" and "width."

1. The pupils may use the reverse sides of the rectangles used in section 1 of Part A. The teacher should explain that "length" is the longer of the two sides, and that "width" is the shorter of the two sides. As in the procedure of Part A, she should make clear that the first letters—"*l*" and "*w*"—are used to replace words. A brief summary of the explanation and the forms of the shorthand should be written on the blackboard and on the pupils' papers. (See Part B of the chart.)

2. Rectangles of different sizes should be drawn on the blackboard and on paper by the pupils and the letters "*l*" and "*w*" should be written in the proper places.

3. Examine and discuss Part B of the chart.

Part C, Purpose: To teach the arithmetic shorthand for "side."

1. Give each pupil a square made of construction paper and tell him that in the case of the square, each side is called "side" and that it has no other special name as do the sides of the triangle. The letter "*s*" will be the shorthand form that the pupils will originate and it will, of course, be the acceptable one. This letter should then be made to appear along the squares along the sides.

2. The blackboard procedure for the square should follow that described in Section 2 of Part B.

3. The class should study and discuss Part C of the chart.

Part D, Purpose: To teach the fact that when letters appear side by side or when a number precedes a letter without any sign appearing (in arithmetic problem) the process indicated is multiplication.

1. The teacher should refer to Part A and to the fact that "*b*" represents "base" and "*h*" represents "height." Her explanation might be worded in this manner:

> "If one wishes to show the multiplication of base times height, he may write it this way (he writes it in words on the blackboard) or he may write it this way (he writes it in shorthand form). If you saw that in some other class, you might try to pronounce it, but in arithmetic class we are expected to multiply the lengths of the lines those letters represent. If one wishes to show the multiplication of length by width, he may write it in words (he does so) or he may write it in shorthand (he does so). If one wishes to show the multiplication of 4 times the side of a square, he might write it in words (this he does) or he might write it in arithmetic shorthand (this he does). If you see "*ab*" in arithmetic class, you will know that it means the length of "*a*" times the length of "*b*," but if two letters are together in another class, we usually pronounce them as a word."

2. Pupils should note Part D and discuss it.

Part E, Purpose: To reteach the meanings of the terms circumference, diameter, and radius, and to teach that "*c*," "*d*," and "*r*," are the shorthand for circumference, diameter, and radius, respectively.

1. Give each pupil a cardboard circle and three strings the lengths of the three principal measurements of the circle. Names of the three measurements should be given the pupils if they have forgotten them. The pupils are to try to fit these three strings along the proper lengths of the circles. The initials of the three words will seem to be the natural shorthand form. These initials should be used on the cardboard circles.

2. On the blackboard, pupils should repeat the technique used for the cardboard circles.

3. Pupils should study and discuss Part E of the chart.

Introducing Mr. "0" and Mr. "Decimal Point"

Richard Ernst
Gainesville, Florida

Probably the biggest problem that faces 7th grade teachers is to find some way to stimulate the pupils' interest as well as aid his learning of a subject. I was faced with this problem while interning in a junior high school. My directing teacher suggested I teach the unit on decimals. He warned me that 7th grade students were often afraid of decimals and as a result of this, interest was often low. I took this warning to heart and made up my mind to find a way to introduce the unit in such a manner as to create interest and also to teach place value and the use of the decimal point.

With the help of Dr. K. L. Kidd, my subject coordinator at the University of Florida, I developed a card system which seemed to serve both purposes. I am aware that perhaps other teachers have used an idea similar to this. However, this card system was very helpful to me and was considered a very effective teaching aid by my directing teacher. Because of this, I decided I would like to share it with other teachers.

The cards I used were 8 inches by 10 inches and were made of heavy white poster paper. These cards had either a number from 1 to 9, a zero, or a decimal point painted on them in large bright blue lettering.

Now for a brief summary of how I used these cards. I divided the class into two equal sections. One section went to the front of the room and the other remained at their desks. I gave each one of the students at the front of the room a card. In order to have everyone holding a card, I made several zeros. I introduced the cards as a contest between the pupils at the front of the room and those at their desks. If the pupils up front could form, in ten seconds, a number that I called out, they would gain one point for their side. If they failed to form the number, it would be a point for those at their desks.

Before starting the contest, I had each pupil with a card announce how much he was worth. This was easy until we came to the zeros. They decided they were worth nothing. The decimal point said he didn't have any value at all.

After this, we started the contest. At first I called simple numbers such as 69 and 135. The pupils formed the number by stepping forward and taking up their proper place in the formation of the number, holding the card in front of their chest. When they became accustomed to forming the numbers, I began calling out decimal fractions such as 6.1 and 73.89. At this point we stopped and talked about the decimal point. What does it do? Does it have any value? Does it hold a place such as units' or tens'? By referring to the numbers formed by the cards, I pointed out that the decimal point was similar to a sign post. It merely separated the whole number from the fractional part of the decimal fraction. Also, the pupils were able to see that the decimal point did not have any value and that it didn't hold a place such as units.

Upon completion of the discussion of the decimal point, we again formed some more numbers, this time using the zeros. I called out such numbers as 2.06 and 31.0009. It took some time before the pupils could associate hundredths, ten-thousandths, etc. with the proper number

of places past the decimal point. However, after some practice, and with the help of the contest speeding along the desire to learn, both the pupils with the cards and those at their desks became proficient in knowing that when I called out "two hundred thirty-one and eight ten-thousandths" there would be three places in front of the decimal point and four places in back of the decimal point.

When the pupils could form numbers using the zero, I posed such questions as these: What does the zero do? Does it have a value? Does it function like the decimal point? The pupils agreed that the zero didn't have any value and that it had a different function from the decimal point. The question we talked quite a lot about was, "What does the zero do?" I had the pupils form a number such as 6.003. Then I asked the two zeros to drop out. What happened to the number? Of course everybody saw that it changed the number. It was 6.3 instead of 6.003. From this it was relatively simple to show that the 0 was merely a place-holder. In the number I gave, I said there were no tenths so we put in a zero to hold that place and the same holds true for hundredths.

The pupils seemed to have gained a better understanding of place value, the function of the zero, and the function of the decimal point through the use of these cards and through intellectualizing and verbalizing the ideas relative to the function of the zero and the decimal point. Also, it wasn't hard to see that they really enjoyed using the cards, especially when I made a contest out of it. Interest in decimals certainly appeared to be on the way up.

A Game of Squares

GEORGE JANICKI
Elm School, Elmwood Park, Ill.

THE pattern above suggests a game of dots to any bright student. In my classes, I used this puzzle as follows:

First set a time limit of one minute. (This makes it very exciting!)

The idea is to start at any place and to connect each dot, and to try to complete as many whole squares as possible.

You do not permit any diagonal connections. NEVER TAKE PENCIL OFF THE PAPER!

You cannot retrace or cross any previous line.

When you end up in a blind alley, the game is finished.

The scoring is: 1 point for a completed square; 3/4 point for 3 sides of a square completed; 1/2 point for two sides completed; and 1 point for one side since you might possibly have such an arrangement.

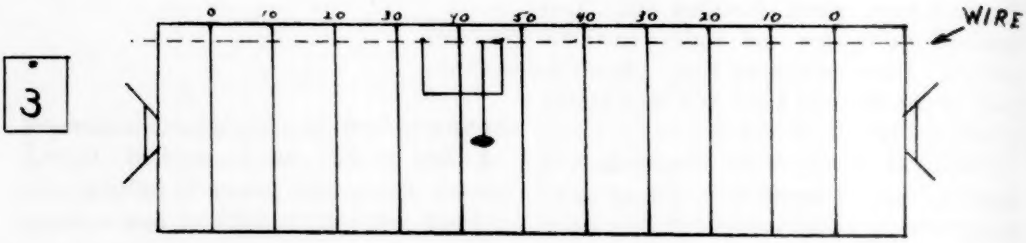
The average score is: 7 points; the real bright students reach 8 3/4. This game is really fascinating.

I recommend it highly to any arithmetic teacher in grades 4, 5, 6, 7, 8.

(You can change the scoring points for lower grade students to all whole numbers: 4 points, 3 points, 2 points, and one point since they may be unfamiliar with adding unlike fractions. In such cases, 48 is top score; see how high they can make using these scoring rules.)

Arithmetic Football

Earl A. Karau
Saginaw, Michigan



Arithmetic football is a game of mental arithmetic. It is played without the aid of pencil or paper. A football field made of roll paper or oilcloth is laid out as shown in the drawing. It is 120 inches long and 36 inches wide. The line markers are ten inches apart for easy layout of the field.

Over the top edge of the field a wire is strung with marks one inch apart. Put a small dent at every inch mark so it is easy to set the ball wire in place. One football on a moveable wire is used. The wire should be at least twelve inches long and the size of the football should be about two inches by three-quarters of an inch. This ball moves up and down the field as each team plays the part of the offensive team.

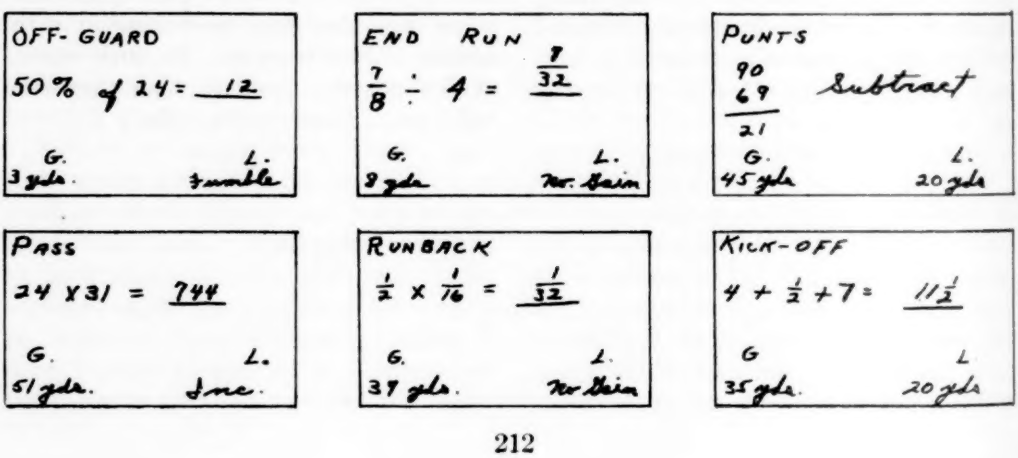
Four "down" cards are used in place of a "down" box. You hang up the number of the "down" so the quarterback will know what play to call. A wire ten inches long is used like the chain measure to mark off the ten yards to a first "down." This hangs from the top wire and must be moveable. It is placed and used as in an actual game. It shows how many yards are needed for a first "down."

You will need enough problem play cards to carry on the game. The cards are put in piles according to the type of play thereon. The following types of plays were found suitable:

		Gain
Off-guard	(Easiest problems)	1 to 3 yds.
Off-tackle	(Easy problems)	1 to 8 yds.
Reverse	(Easy problems)	1 to 25 yds.
End run	(Easy problems)	1 to 15 yds.
Pass	(Harder problems)	1 to 60 yds.

Punts
Field goals
Kick-off
Runbacks

Here are some examples of plays and their problems:



Each card carries a gain or a loss. If the problem is answered correctly within the 30 second time limit you use the gain section. If the time runs out or the wrong answer is given you use the loss section. The loss may be: no yards gained, yards loss, fumble, or in case of a pass, it may be incomplete or intercepted. The cards are turned over, one at a time, so even the teacher doesn't know what problem is coming up next in any of the play stacks.

A stop watch is fine for timing but any sweep-second-hand watch will do. A fair knowledge of football is needed by the teacher as he acts as referee of the game.

Two teams are picked and a quarterback from each team is elected to run the team. The quarterback should know a little about football as he calls all the plays including the person to carry the ball. The teacher flips a coin to see which team shall kick-off. If team *A* kicks-off the moveable football is placed on the *A*'s forty-yard line. A kick-off card is picked up by the teacher and quarterback *A* picks a team member to answer the problem on the card. The problem is given, and if answered by the player, the ball is moved down the field the number of yards shown on the card. Now team *B* has the chance to run the kick-off back so the *B* quarterback picks a team member to answer the problem the teacher has picked up off the top of the run-back stack. The player *B* has 30 seconds to answer the problem. If he answers the problem correctly the ball is moved the number of yards shown on the card, but if he misses the problem the loss may be: ball down at that point or fumble, depending on the card.

Team *B* now has the ball and the moveable ten yard marker is placed so that one end is at the point of the ball and the other end is ten yards away. This is the same principle as the chain measure used at a real game. "Down" 1 is put up on a hook and team *B* is ready to try its first play of the game. Quarterback *B* picks a team player and the type of play he wants him to run. The quarterback can't call on one team member again till every one on his team has had a chance to carry the ball. The quarterback has five different types of plays he may call: off-guard, off-tackle, end-run, reverse, and pass. The easier problems give less yards on a play. The pass plays are the hardest but the greatest gains are made on this type of play. On each card is written the number of yards to gain if answered correctly and the card also carries the loss if the wrong answer is given. The teacher makes up his own problems and adds new problems as new work is learned in class. You can fit your cards to any grade level you wish. For seventh grade it is best to have more than a hundred play cards and use them over again. The students work outside of class to learn what types of problems are most difficult for them. They work on their weak points so as not to let their team down.

Field goal cards are used for the extra point when a touchdown is made. You may also let a team try a field goal if within the twenty-yard line for these cards the point is made if the problem is answered correctly and the point is missed if the problem is missed.

Penalties can also be applied, for example, if one team member helps another team member it is fifteen-yards and the "down" remains the same. If team *A* bothers team *B* by talking, team *A* is penalized five yards and the "down" remains the same. Not many penalties are marked off as each team really wants to win. The teacher may set any time limit he wishes for the duration of the game.

The use of this game in seventh and eighth grade classes has been highly successful in stimulating rapid and accurate computation without pencil and paper. Both the boys and girls enjoyed it and worked hard to improve their skill in mental arithmetic.

Arithmetic in the Child's Future*

Joseph J. Urbancek
Chicago Teachers College, Chicago, Ill.

This morning we shall talk about ARITHMETIC IN THE CHILD'S FUTURE and what it shall mean to the men and women of the future who presently are today's children. Indeed adults have a stake in this future too, be they teachers, parents or relatives, because possible new developments may well affect their own lives.

The manner in which arithmetic is taught and the interest that teachers can engender in children will determine to a large extent how much mathematics and science the more capable children will pursue in their high school and college careers. For many of them, their professions and careers are determined and shaped by well informed and inspired teachers who make the arithmetic, science and mathematics they are teaching appear so easy, logical, interesting, absorbing and stimulating to further study. Thousands of leaders today can point to a time in their elementary school lives when what the teacher was teaching, or the added plus that she gave what she was teaching set off a spark in their young minds that first led to a determination to know more about it, and ultimately to a career and leadership; in some cases to world renown. The guidance that the good teacher gives at such a time, the know how is all important.

In our schools today are the great minds of the future. Who knows which teacher or how many teachers will stimulate into action new intellectual giants who will probe realms of human activity to come? It is the elementary teacher who lays the foundation for a lasting interest in mathematics and thus sends youngsters on the road to becoming the specialists we need so urgently.

There are many avenues to travel in the fields of science and manifold applications from which all humans may benefit. In each avenue traveled there are many levels of attainment due to individual differences, but we have need for the services of all. Despite the many branches of science, their numerous applications, and the many levels at which individuals may work—there is one common bond. It is an important and effective thread that runs through all of them regardless of the level of operation. That bond is mathematics and the most important base of it is arithmetic. In a modern world it is impossible to function without mathematics and that is why teaching arithmetic well is so important.

The achievement of men and women who have made use of science and mathematics have more than once completely revolutionized the world. Whole areas of the world have been changed—old jobs lost and new and better ones created as a result of the new industries developed.

Science and Mathematics Work Together

Lists of important men and women who in past decades have made their contributions to scientific and social development would fill a very large book, and stories about them and their contributions would fill volumes. The essential ingredient common to the start of their success was a command of the fundamental processes of arithmetic and basic science. So will it be in the future of our present children.

It may well be true that too many

* Excerpts from a radio broadcast in the Chicago Teachers College series.

people take things for granted but perhaps it is more likely that they do not see the connection between basic knowledge, progressive development, and ultimate application.

We are living in the jet and atomic age. Metals like titanium, lithium, vanadium, zirconium, uranium, plutonium, thorium, and beryllium are some that may well have an influence on your life and surely on the lives of our future citizens. Until more was known of these metals and how to use them, progress on rockets, jet engines and guided missiles was slowed, chiefly because previously known metals could not withstand the high temperatures and other damaging effects produced by the modern engines and missiles. Recent research problems, of the many with jet planes included "crashing the sound barrier" and cooling the "skin" of the newest jets. Many aspects of both problems have been solved. When a plane is traveling above the speed of sound its "skin" heats up several hundred degrees. The pilot inside could not survive such a condition were it not for a refrigeration system containing a turbine weighing approximately two pounds. This turbine rotating at 100,000 revolutions per minute cools to 40° the air fed into the plane's cabin and does this in 2/10 of a second.

It should be easy to understand that with most of these new developments of forward progress, thousands of problems were encountered, and solved, and additional problems were created. It is axiomatic, but necessary, to note that the preliminary educational training needed in the elementary school was in arithmetic and the basic sciences which led later to the blooming and fruition of full fledged scientists and technicians.

One of the shrewdest investors of our time has predicted that there could be a shortage of uranium within ten years or less. Scientists and others have predicted an exhaustion of oil and gas supplies within one-half century. Still others predict harnessing the sun for solar energy. If these predictions come true, and some of them may well do so, it is clear that the human race will not lack problems in the future—and equally clear that our more able children will need to be well grounded in arithmetic and science.

One company in the electronics industry employing more than 2000 scientists and engineers discusses "The World's Most Promising Technological Revolution" in which the future of the field of electronics seems most promising. Already the many companies in this industry are doing an annual volume of business reaching 10 billion dollars. This fabulous figure stated another way is 10,000 million dollars, and it is expected to double by 1965, less than ten years from now. Need more be said about the importance of building adequately the background of our children in arithmetic and science. By the time these children have completed their education for careers, doubtless many new developments in electronics will involve many new problems on which their training and ingenuity can go to work. To get some estimate of the future, one needs to reflect on the fact that currently more than 5 billion dollars per year are being spent on research for industry in developing and expanding the uses of frozen foods, electronics, wonder drugs, synthetic rubber, test tube fabrics, and hosts of others.

Our news media often carry information about current experiments in man made satellites, new jet planes, guided missiles, and flights across the United States in 20 to 25 minutes at 10,000 miles per hour at heights of 100 miles above the earth. Obviously we shall need our children well trained to become the scientists of tomorrow—when they deal in problems of nuclear, thermonuclear, ion, and photon propulsion.

Besides the great need we have for scientists, engineers and technicians, imagine the additional international diplomatic good they can perform when they represent us in all parts of the world demonstrating the effectiveness of our way

of life. We need more of them, good ones, and cannot have them too soon.

What must be done about our health problems? There are many to solve. It is estimated that the losses due to the common cold alone annually run to 5 billion dollars. Whoever solves this big sniffle problem will doubtless receive the plaudits and blessings of millions.

Finally arithmetic may well teach: truth and honesty, confidence and dependence, order and sequence, interrelations between quantities, respect for its value in the activities of the human race, the basis for courses in higher mathematics and science which frequently open the door to the unknown thus giving us a better way in which to live.

EDITOR'S NOTE. All good teachers are cognizant of the role they play in stimulating and encouraging boys and girls who reveal curiosity and of the satisfactions that come to the teacher when a pupil gives evidence of insight. They are also aware of the frustrations that come to their pupils in school and in later life when they cannot do simple sums accurately and think in terms of numbers and quantity. And it is sad to note that many marriages are in jeopardy because one or both parties cannot with responsibility think and plan with the family finances. Truly, the teacher of arithmetic has a very great responsibility.

The Seventeenth Christmas Meeting of the National Council of Teachers of Mathematics

Arkansas State College, State College, Ark. December 26-29, 1956

There will be a full program of discussions on arithmetic. The program will be printed in the December issue of *The Arithmetic Teacher*.

Report of the Membership Committee

National Council of Teachers of Mathematics

MARY C. ROGERS, CHAIRMAN
Roosevelt Junior High School, Westfield, New Jersey

The Membership Committee of the National Council of Teachers of Mathematics is pleased to report to you the results of the 1954–1956 Membership Campaign, and to thank you for your fine support and direct helpfulness in bringing about the generous increase in membership which was accomplished. It was this assistance on the part of each of you as individuals which has largely made this accomplishment possible. We are indeed grateful to you for your enthusiastic cooperation.

Record of Membership Growth

We are submitting herewith the latest membership analysis based on the membership count by states received from the

Washington Office early in May, 1956. Please study it carefully for the over-all picture it presents, and for the results which you have helped bring about in your various states.

In this analysis, you will notice a total count of 13,827 members. This represents a relative achievement of 92.2% of the total 15,000 goal set for us in April, 1954.

More specifically, we are pleased to announce:

1. 56% of all states and regions show a relative achievement of 90%–188% of their established goals.
2. 13% have attained from 86%–89% relative achievement.
3. 12% have a standing of 80%–84% of goal.
4. 13 states have "gone over the top" while 14 states have fewer than 20 members each to secure to reach their goals.

Plans for the Future

This report of membership progress was reported to the National Council Board of Directors at its 1956 Summer Meeting in Los Angeles. Your Membership Committee was asked to continue its work with you for another two year period. Our goal for total membership remains 15,000; suggested goals for states and regions will remain as heretofore established.

Most commendable growth in NCTM membership has been made, but the services of the Council still are not reaching a great many mathematics teachers and other persons throughout the country who are interested in the betterment of Mathematics Education from kindergarten through college. We want very much to correct this inadequacy, and we shall continue to work hard to do so. With your increasingly enthusiastic support and assistance we are confident we shall make great strides toward reaching these people.

How can you help membership growth in your state? How can you further NCTM membership progress toward the 15,000 goal? May we suggest that future procedures follow closely those you have found effective in the past.

1. Will each of you please secure at least *one new member* for the Council. We have found the "Each One Win One" technique to be one of our most valuable aids and urge its continuance and positive use by *all current leaders and members.*

2. *Prompt renewals* of present memberships greatly facilitate the keeping of records and the preparation of frequent reports.

3. You, who are members of the Mathematics Staff in Colleges of Education and similar education centers, can be of invaluable assistance through *your continued stimulation of interest* in NCTM services among your students.

4. A similar service can be rendered with increasingly fine results by you who are *supervisors or department chairmen* in your work with your teachers.

5. *Many State and other local associations* of mathematics teachers are affording the Council most valuable publicity at their meetings and through publications. A continuation and expansion of this fine support will be deeply appreciated.

6. Your continued strong support of NCTM *State Representatives* will greatly facilitate the outstanding work they are performing for the Council.

7. May we remind you that Library and other institutional subscriptions are counted toward the 15,000 goal although they are not considered as memberships.

Your Membership Committee will keep you informed of membership progress as often as this is available to us. You will hear from us at least twice a year through this publication. Reports will be made at all National Council meetings and will be available for release at your local meetings and through your local publications. We welcome your advice and suggestions.

Analysis of Membership Growth: April, 1955 to May, 1956

States	April 1955	May 1956	Goals	Per Cents
1. Alabama	108	135	170	79%
2. Arizona	40	106	60	177%
3. Arkansas	92	106	171	62%
4. California	579	888	879	101%
5. Colorado	143	168	180	93%
6. Connecticut	180	220	228	96%
7. Delaware	45	66	71	93%
8. District of Columbia	131	214	185	116%
9. Florida	237	346	351	99%
10. Georgia	145	152	186	82%
11. Idaho	12	12	18	67%
12. Illinois	764	1,075	1,188	90%
13. Indiana	352	478	542	88%
14. Iowa	202	270	296	91%
15. Kansas	217	285	311	92%
16. Kentucky	73	98	123	80%
17. Louisiana	176	236	276	86%
18. Maine	52	67	69	97%
19. Maryland	207	269	285	94%
20. Massachusetts	323	389	440	89%
21. Michigan	400	544	600	91%
22. Minnesota	246	311	391	80%
23. Mississippi	83	106	123	86%
24. Missouri	238	278	330	84%
25. Montana	50	59	60	98%
26. Nebraska	105	126	165	76%
27. Nevada	17	23	14	164%
28. New Hampshire	44	74	65	114%
29. New Jersey	366	511	561	91%
30. New Mexico	61	79	89	89%
31. New York	863	1,152	1,181	98%
32. North Carolina	172	198	255	78%
33. North Dakota	31	28	44	64%
34. Ohio	480	613	740	83%
35. Oklahoma	163	212	237	89%
36. Oregon	109	176	134	131%
37. Pennsylvania	655	844	920	92%
38. Rhode Island	50	52	71	73%
39. South Carolina	83	102	137	74%
40. South Dakota	33	52	36	145%
41. Tennessee	154	207	246	84%
42. Texas	480	673	672	100%
43. Utah	43	60	48	125%
44. Vermont	25	41	41	100%
45. Virginia	250	310	357	87%
46. Washington	167	226	228	99%
47. West Virginia	77	77	188	41%
48. Wisconsin	307	465	420	111%
49. Wyoming	27	35	36	97%
	9,867	13,214	14,388	92%
Hawaii		37		
Other U. S. Possessions	43	102	74	188%
Canada	142	233	201	116%
Foreign	214	241	339	71%
GRAND TOTALS	10,256	13,827	15,002	92%

An Arithmetic Spell Down

Did you ever take part in an old-fashioned "spell down?" Do you remember the keen competition among the leaders and how some pupils secretly studied words so they could spell better than their colleagues? In rural schools where several grades occupied the same room, teachers frequently would have pupils from grades 4, 5, and 6 all in the same competition. Perhaps the competition was psychologically bad for some but it did stimulate many others to higher achievement. Many teachers have adapted the "spell down" technique to other areas such as arithmetic. The procedure may be changed but there is great value in the oral situation because that is similar to many of the uses of arithmetic by people outside of school. Exercises should be so selected that the computation is a minimum and can be completed without paper and pencil. Below are some exercises that can be used in grades 4, 5, and 6, and even higher. Teachers are sometimes amazed at the work that certain pupils can do mentally even before they have formally studied this in school.

1. How much is $4 \times 0 + 5$?
2. Grade 4 has 28 pupils and grade five has 30. On Tuesday there were 3 absences in each grade. Which grade had the better attendance record?
3. How much time is there from 15 minutes past 10 to 5 minutes before 11?
4. What number is third when you count backwards from 201?
5. How many half-pints equal one gallon?
6. When a third of a pound of butter lasts us 3 days, how long should the remainder of the pound last?
7. At 60 cents per hour for work, how much is 12 minutes worth?
8. If the first of May is on Tuesday, on what day of the week will the tenth of May fall?
9. My watch loses 6 minutes per day. At what time should I set it at 8 in the evening so that it will show correct time at 8 in the morning?
10. If 1000 sheets of paper form a pack 4 inches thick, how thick is each sheet?
11. Pete missed the first 3 times when he tried to toss the ball into the basket but he succeeded the next three times. What is his average of successes?
12. Which is the longer: 2.5 ft. or 2 ft. 5 1/2 in.?
13. Which is the largest fraction: 7/8 or 19/20 or 2/3?
14. When a price is 4 for 49¢, how much is saved by buying at the price for four rather than buying 4 items one-at-a-time?
15. When 8 inches of snow equal one inch of rain, how many inches of rain will 12 inches of snow equal?
16. How many square feet are there in a square which is 1/2 foot on each side?
17. When Mrs. Jones buys sirloin steak she realizes that 2/5 of the weight is bone and other non-meat. What fraction is meat?
18. If Mrs. Jones pays 90 cents per pound for the sirloin and it is 2/5 waste, then how much is she paying per pound for the meat part?
19. Is 3/8 of a cupful nearer to 1/3 or 1/2 cupful?
20. If I must allow 20 minutes per pound for roasting meat, at what time in the morning should I place a 12-pound roast in the oven so that it will be done at 1:30 P.M.?

Garden of Mathematics

DINAH STARR II
Cambridge, Mass.

Janie's homework had arrived. She had a bad cold. She drew a deep breath and opened her Math book. "Eighty-seven inches by forty-nine inches by fourteen inches," she muttered. The numbers spun around in her head. Slowly the book slipped from her lap and she fell asleep.

She found herself on a tape measure road and as she walked down it the numbers got up and followed her. The tape ran under an archway and into a garden. A banner hung from the great arch with the words, "The Garden of Mathematics," written in silver upon it. A fountain of liquid measure played in the center of the garden. Around it, four marble pools formed a square around it. Nearby, Roman numerals were figuring out the volumes of lazy prisms taking sunbaths.

On the right side of the garden, the first six multiplication tables of white marble stood in a row against the surrounding hedge of dark green. On the left side were the other six tables.

The dividends and quotients were all at the right tables with the divisors at the head. Along the edge of the garden in between the tables there were flowers of various geometric shapes. In these beds, little prisms played hide-and-seek among the flowers of their shapes.

Opposite the fountain at the end of the garden was a series of steps. On each step was a small throne. Janie looked at the first two. The words "Arithmetic" and "Geometry" were inscribed on the backs of the thrones. She mounted the first two steps and sat down on the chair marked "Geometry." Just at that moment a greenish-yellow two ran past the bottom in hot pursuit of a scarlet four.

Janie awoke with a start and began her Math with a new vigor. From then on, she took pleasure in the numbers, thinking of every one as a character in the "Garden of Mathematics."

The Try-Angle Puzzle

GEORGE JANICKI
Elm School, Elmwood Park, Ill.

THE above pattern of dots suggests the idea that they are to be connected. The answer is yes, but do not take your pencil or chalk off the pattern, and do not go back or retrace any line.

When you complete the pattern successfully, you win and you score 16 points.

If you fail, and fall into a trap, the game ends.

Your scoring is: 1 point for a complete triangle; 2/3 point for 2 completed sides; and 1/3 point for one side of any triangle.

In lower grades, you may want to score in terms of whole numbers; all 3 points per triangle completed, 2 points and one point for partially completed triangles.

I found the above pattern to be stimulating thinking.

It looks easy and is a challenge to critical thinking for all students. It is fun and it is arithmetic!

The Editor Asks

We have many thousands of very good teachers in the elementary schools of this country. Many of these teachers have found certain procedures very helpful in the understanding of arithmetic by their pupils. The editor would like to have accounts of good school practice so that he can relay the information to others. We are all concerned in teaching a type of arithmetic that will enable our pupils to become really intelligent both in their point of view toward the subject and in their understanding and use thereof. Let us share our good experiences.